Graphics (INFOGR), 2018-19, Block IV, lecture 8

Deb Panja

Today: Matrices and introduction to transformations

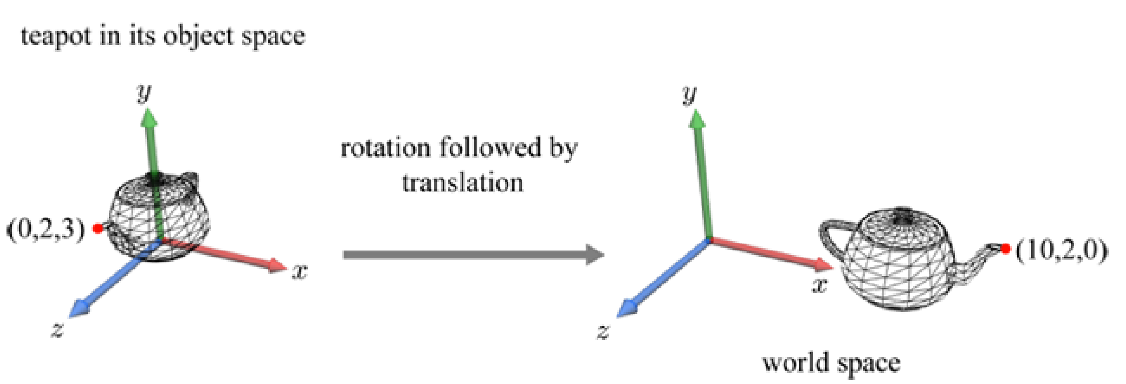
Welcome back!

Today

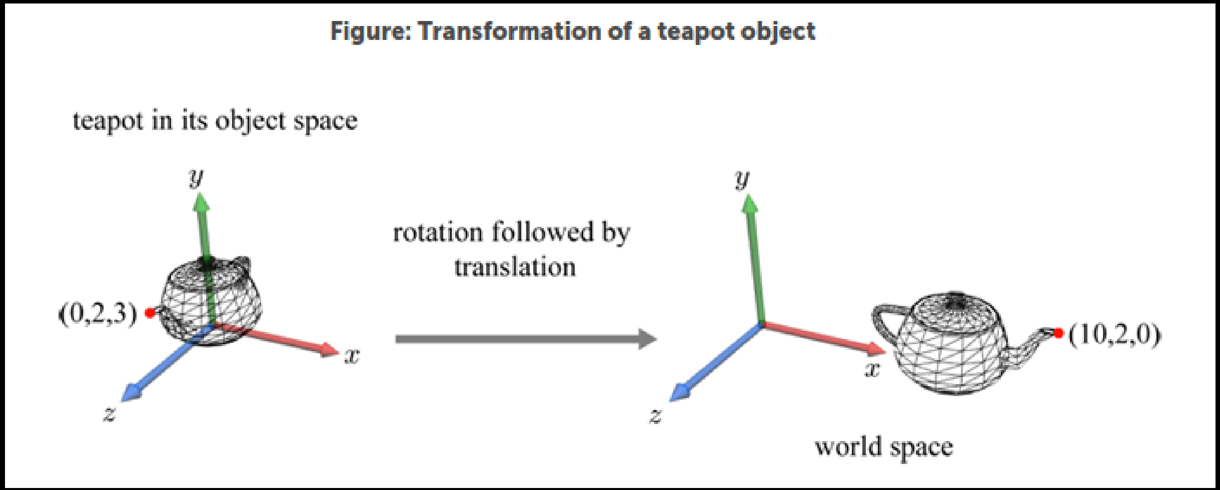
- Matrices: why and what?
- Matrix operations
- Determinants
- Adjoint/adjugate and inverse of matrices
- Geometric interpretation of determinants
- Introduction to transformations

Spatial transformations – part II of the course

Figure: Transformation of a teapot object

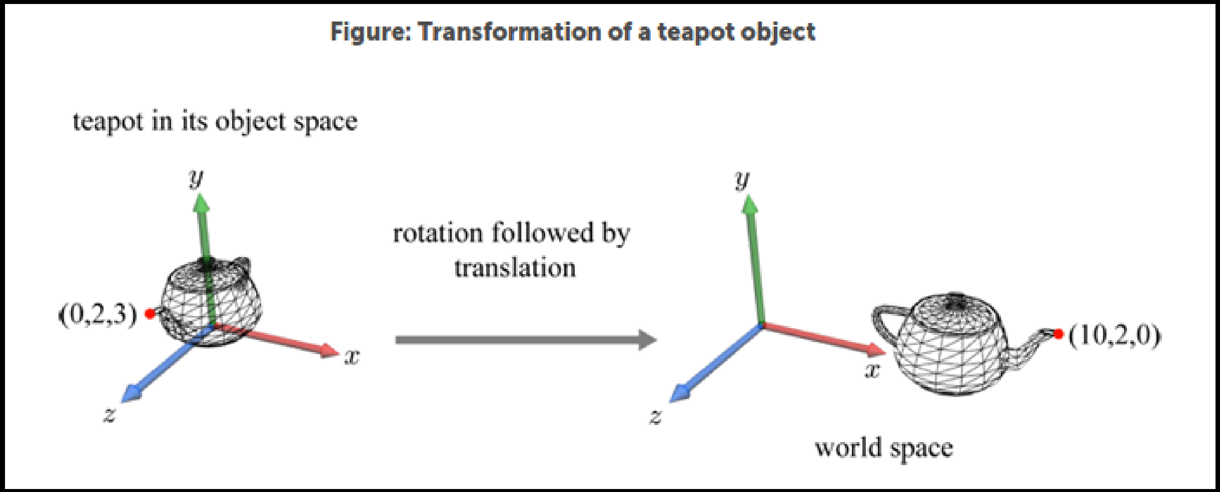


Spatial transformations – part II of the course



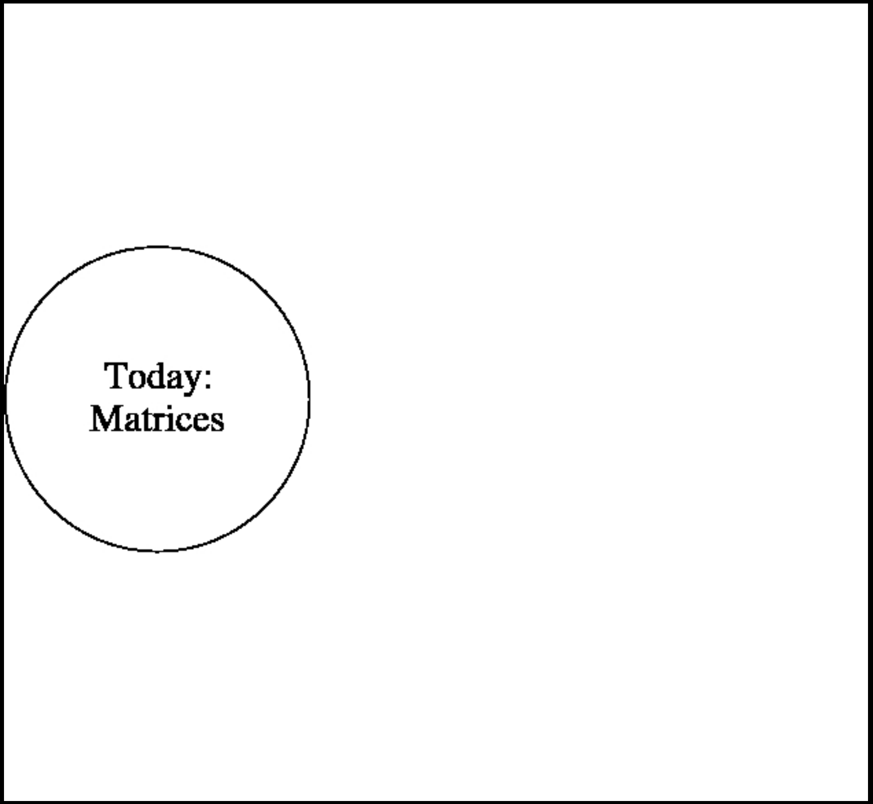
- Why matrices?
  - you need to execute such spatial transformations (a lot!)
  - matrices are the vehicles you need for these tasks

Spatial transformations – part II of the course

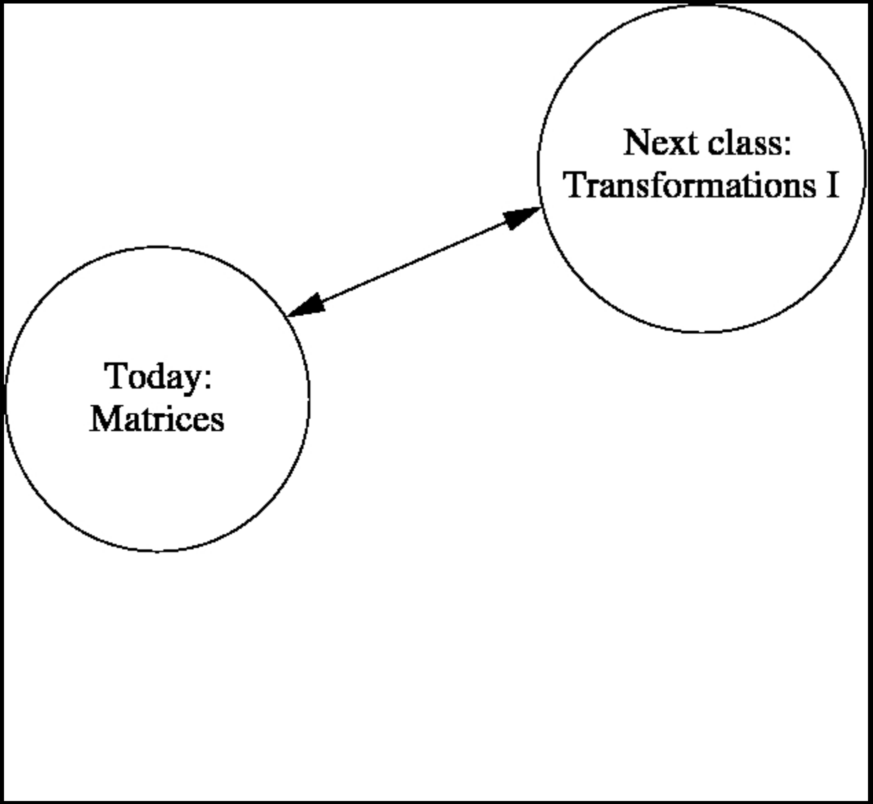


- Why matrices?
  - you need to execute such spatial transformations (a lot!)
  - matrices are the vehicles you need for these tasks
- That means: it's nearly impossible to dissociate matrices from transformations they achieve

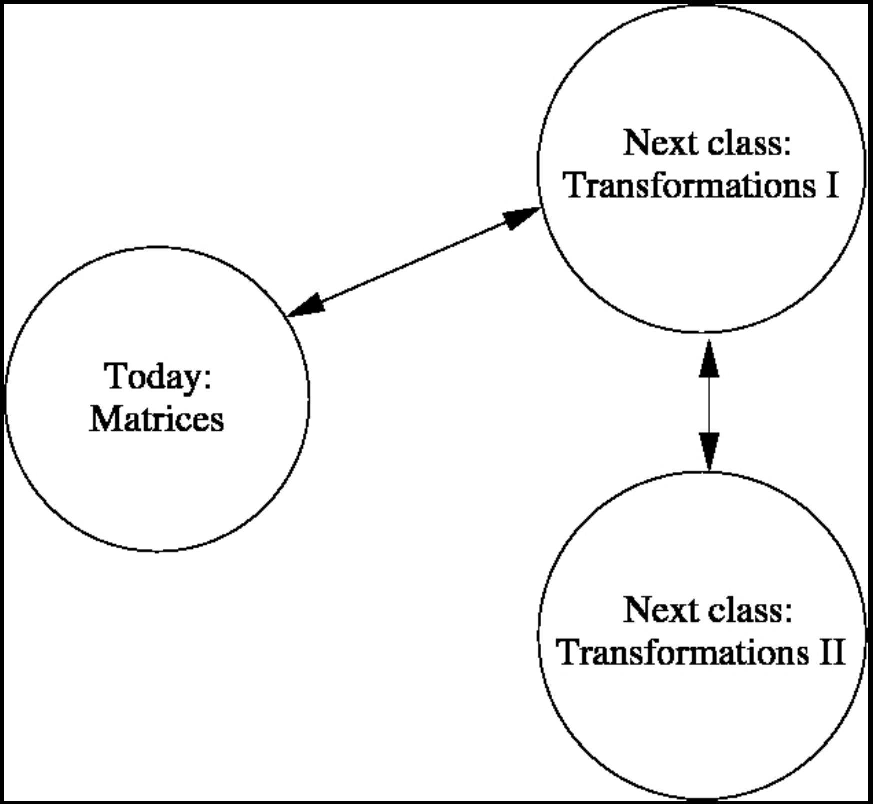
Bigger scheme of things: three upcoming maths lectures



Bigger scheme of things: three upcoming maths lectures



Bigger scheme of things: three upcoming maths lectures



- It's been a choice to make a clean separation this way!

What is a matrix?

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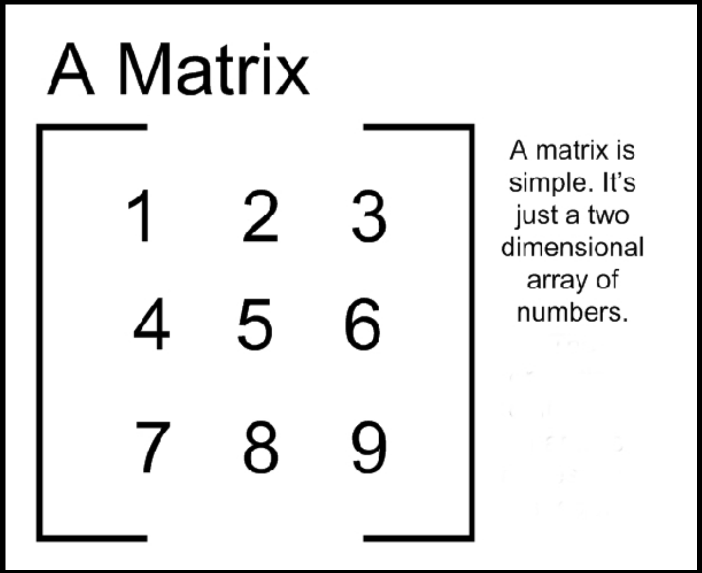
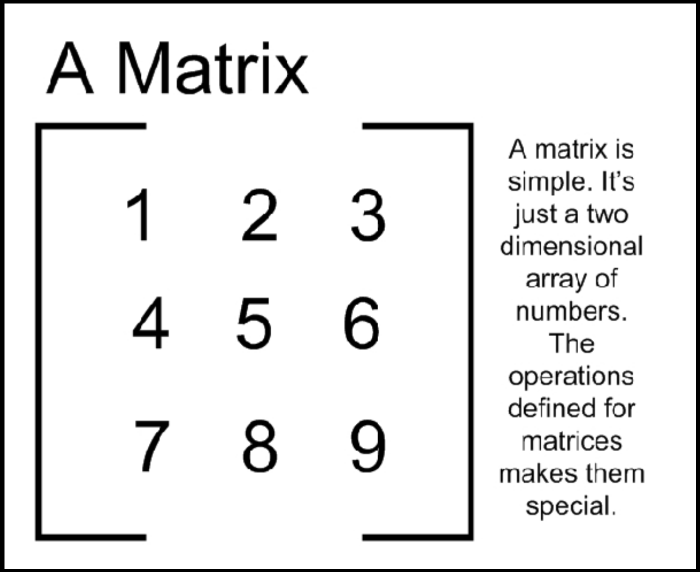
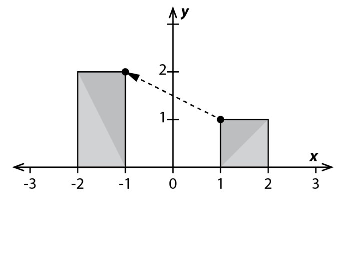


- You'll store it on a computer as a two-dimensional array: arr[3][3]

Why matrices?



- Example: 

(more about that in the next lecture)

Matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \leftarrow i\text{th row of } \mathbf{A}$$

$\uparrow$ jth column of $\mathbf{A}$

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- Dimension of the above matrix: $m \times n$
  - when $m = n$, the matrix is called a square matrix
  - a_{ij} $(i \in [1, m], j \in [1, n])$ are the matrix elements/coefficients
  - shorthand notation $A = \{a_{ij}\}$

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↑ jth column of $\mathbf{A}$

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  - an $m \times n$ matrix: n vertical concatenation of m-dimensional vectors

Matrices

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- A d-dimensional vector is a $d \times 1$ matrix
  - an $m \times n$ matrix: n vertical concatenation of m-dimensional vectors
- A scalar is a 1×1 matrix

Special matrices

- Diagonal matrix: square matrix with $a_{ij} = 0$ for $i \neq j$; e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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- Null matrix (denoted by $\emptyset$ or O): $a_{ij} = 0$, e.g.,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix operations

Addition of matrices

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

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$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+9 & 2+8 & 3+7 \\ 4+6 & 5+5 & 6+4 \\ 7+3 & 8+2 & 9+1 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$$

- Can only add matrices of same dimensions!

Scalar multiplication of matrices

$$\lambda \mathbf{A} = \lambda \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1m} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda A_{n1} & \lambda A_{n2} & \cdots & \lambda A_{nm} \end{pmatrix}$$

$$2 \cdot \begin{bmatrix} 10 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10 & 2 \cdot 6 \\ 2 \cdot 4 & 2 \cdot 3 \end{bmatrix}$$

Addition and scalar multiplication of matrices

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

1. $A + B = B + A$	Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$	Associative Property of Matrix Addition
3. $(cd)A = c(dA)$	Associative Property of Scalar Multiplication
4. $1A = A$	Scalar Identity
5. $A + O = A$	Additive Identity
6. $c(A + B) = cA + cB$	Distributive Property
7. $(c + d)A = cA + dA$	Distributive Property

Subtraction of matrices

- A and B are matrices of the same dimensions; then $A - B = A + (-1)B$;
  - e.g., $$\begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1-0 & 3-0 \\ 1-7 & 0-5 \\ 1-2 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -6 & -5 \\ -1 & 1 \end{bmatrix}$$

  (can only subtract matrices of same dimensions!)

Matrix multiplication

- A and B are $m \times n$ and $n \times p$ dimensional matrices respectively then $C = AB$ is an $m \times p$-dimensional matrix

Matrix multiplication

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- matrix elements $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

  on a computer: $c_{ij} = 0;$ for $(k = 1;\ k <= n;\ k{+}{+})$ c_{ij} += $a_{ik} b_{kj}$

  e.g.,

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 8 & 9 \\ 2 & 7 & 10 \\ 3 & 6 & 11 \\ 4 & 5 & 12 \end{bmatrix} = \begin{bmatrix} 50 & 94 & 178 \\ 60 & 120 & 220 \end{bmatrix}$$

Vector dot product as a matrix multiplication

- Vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_d \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = [u_1 \ u_2 \ \ \ldots \ \ u_d] \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix} = \vec{v} \cdot \vec{u} = u_1 v_1 + u_2 v_2 + \ldots + u_d v_d$$

Matrix multiplication properties

- In general, $AB \neq BA$! (i.e., they do not commute)

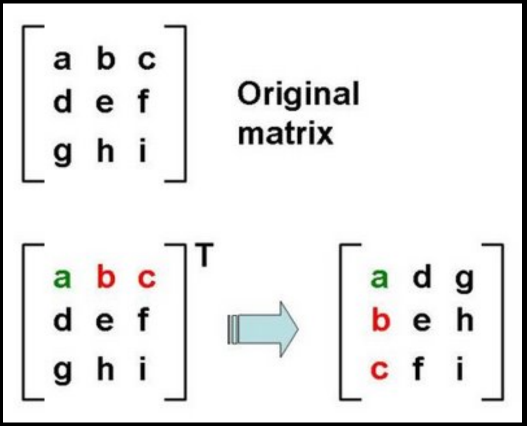
  however, $AB = BA$ if both A and B are diagonal

  in particular, $IA = AI = A$ and $A\emptyset = \emptyset A = \emptyset$

- $A(B + C) = AB + AC$, $(A + B)C = AC + BC$: distributive

- $(AB)C = A(BC)$: associative

Transpose of a matrix



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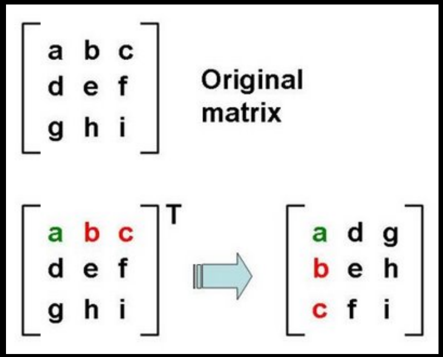

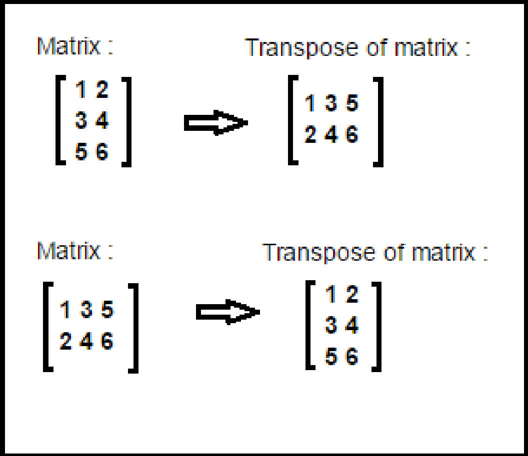

- $A^{\mathrm{T}} = \text{ transpose}(A)$; $(A^{\mathrm{T}})^{\mathrm{T}} = A$

Determinants and cofactors
(only for square matrices!)

Determinants

- Determinant of matrix $A \equiv \det(A)$, or $|A|$; a scalar quantity

- Also as

Notation for 3×3 determinant :

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- Determinants and cofactors are inextricably linked

Determinant (1×1 matrices)

- Determinant of a 1×1 matrix is the value of its element

  e.g., $A = [-5]$, $\det(A) = -5$

Determinant $(2 \times 2$ matrices$)$

- Consider the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det(A) = a_{11}\ \mathrm{cof}(a_{11}) + a_{12}\ \mathrm{cof}(a_{12})$$

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$$= a_{11}\ \mathrm{cof}(a_{11}) + a_{21}\ \mathrm{cof}(a_{21})$$

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  $\det(A) = a_{11}\ \mathrm{cof}(a_{11}) + a_{12}\ \mathrm{cof}(a_{12})$

  $\mathrm{cof}(a_{ij}) = (-1)^{i+j}\ \det(\mathrm{minor}(a_{ij}))$

  $\mathrm{minor}(a_{ij})$ is the matrix without the i-th row and j-th column of A

Q. What is the determinant of the above matrix A?

Determinant (2×2 matrices)

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Q. What is the determinant of of the above matrix A?

A. $\operatorname{cof}(a_{11}) = \det(a_{22}) = a_{22}$; $\operatorname{cof}(a_{12}) = -\det(a_{12}) = -a_{21}$

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$; note also that $\det(A) = \det(A^{\mathrm{T}})$

- Example:

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (2)(3)$$
$$= 4 - 6$$
$$= -2$$

Determinant $(3 \times 3$ matrices$)$

- Consider the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

  $\det(A) = a_{11}\ \mathrm{cof}(a_{11}) + a_{12}\ \mathrm{cof}(a_{12}) + a_{13}\ \mathrm{cof}(a_{13})$

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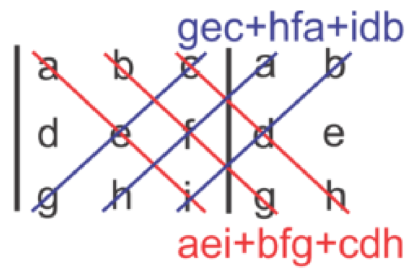
$$\begin{aligned}
\det(A) &= a_{11}\ \mathrm{cof}(a_{11}) + a_{12}\ \mathrm{cof}(a_{12}) + a_{13}\ \mathrm{cof}(a_{13}) \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} \\
&\quad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} = \det(A^{\mathrm{T}})
\end{aligned}$$

- Example:

$$\begin{aligned}
\det\begin{bmatrix} -5 & 0 & -1 \\ 1 & 2 & -1 \\ -3 & 4 & 1 \end{bmatrix} &= -5 \cdot \det\begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix} - (0) \cdot \det\begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} + (-1) \cdot \det\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \\
&= -5[2-(-4)] - 0[1-(3)] - 1[4-(-6)] \\
&= -5(2+4) - 0(1-3) - 1(4+6) \\
&= -5(6) - 0(-2) - 1(10) \\
&= -30 - 0 - 10 \\
&= -40
\end{aligned}$$

Determinant (3×3 matrices), Sarrus' rule



Now, subtract the sums: (aei+bfg+cdh) - (gec+hfa+idb)

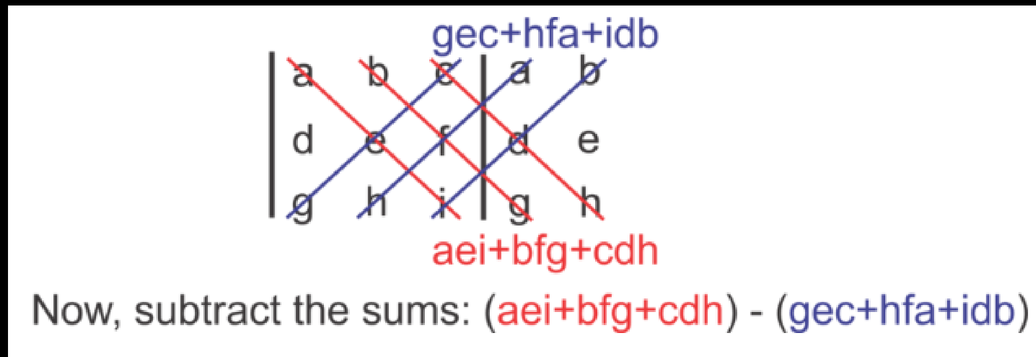
$$\begin{bmatrix} 2 & -3 & 5 \\ -4 & 7 & 1 \\ 3 & 8 & 6 \end{bmatrix} \begin{matrix} 2 & -3 \\ -4 & 7 \\ 3 & 8 \end{matrix}$$

$$(2 \cdot 7 \cdot 6) + (-3 \cdot 1 \cdot 3) + (5 \cdot -4 \cdot 8) = 84 + -9 + -160 = -85$$

$$= (3 \cdot 7 \cdot 5) + (8 \cdot 1 \cdot 2) + (6 \cdot -4 \cdot -3) = 105 + 16 + 72 = 193$$

$$-85 - 193 = -278$$

Determinant (3×3 matrices), Sarrus' rule



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$$= (3 \cdot 7 \cdot 5) + (8 \cdot 1 \cdot 2) + (6 \cdot -4 \cdot -3) = 105 + 16 + 72 = 193$$

$$-85 - 193 = -278$$

- Does not work for 2×2, 4×4, ... matrices

Adjoint/adjugate and inverse
(only for square matrices!)

Cofactor matrix and adjoint/adjugate

- Cofactor matrix $C = \{c_{ij}\}$ of matrix $A = \{a_{ij}\}$
  i.e., $c_{ij} = \text{cof}(a_{ij})$

  example: $$\text{cof}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

- Adjoint/adjugate$(A) \equiv \text{adj}(A) = \text{transpose}(\text{cof(A)})$

Inverse of a matrix

- Inverse of $A \equiv A^{-1} = \dfrac{\mathrm{adj}(A)}{|A|}$
- Has the property $AA^{-1} = A^{-1}A = I$ (identity matrix)
- Inverse does not exist if $|A| = 0$; then A is a singular matrix

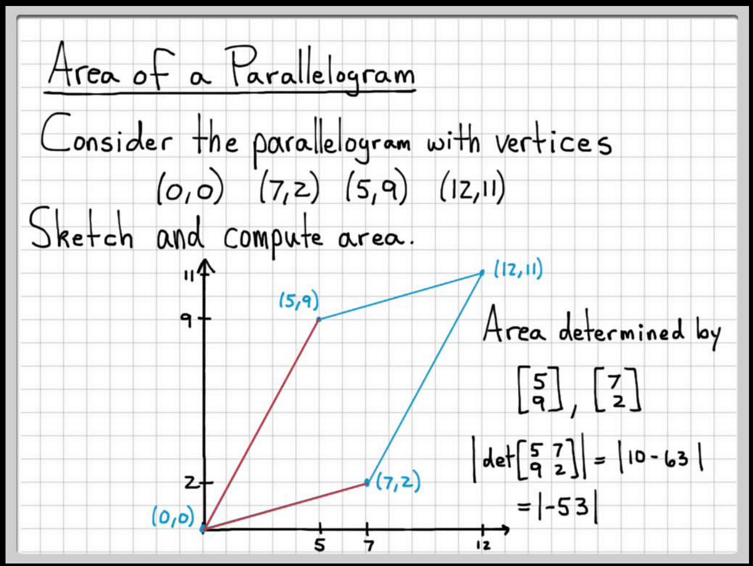
Geometric interpretation of determinants

Determinants of 2×2 matrices

- Consider the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

and the two vectors $\vec{u} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$

then $\det(A)$ is the oriented area of the parallelogram formed by $(\vec{u}, \vec{v})$



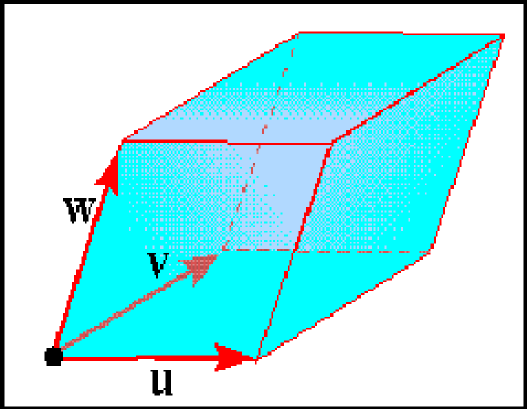
Oriented area is positive if $\vec{u}$ to $\vec{v}$ requires a counterclockwise rotation. Otherwise oriented area is negative.

Determinants of 3 × 3 matrices

- Consider the 3 × 3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

  and the three vectors $\vec{u} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{23} \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$

  then $\det(A)$ is the oriented volume of the parallelepiped formed by $(\vec{u}, \vec{v}, \vec{w})$



Oriented volume is positive if $(\vec{u},\vec{v},\vec{w})$ forms a right-handed co-ordinate system. Otherwise oriented volume is negative.

Determinants of 3×3 matrices

- Consider the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

  and the three vectors $\vec{u} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{23} \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$

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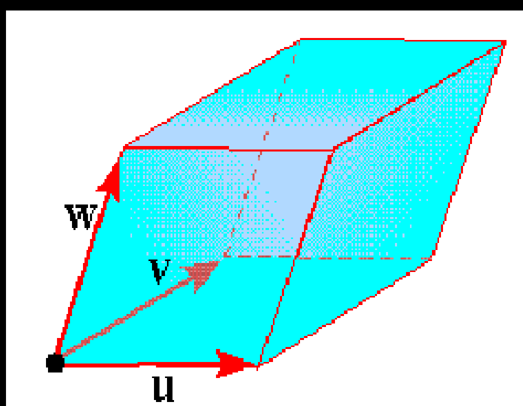
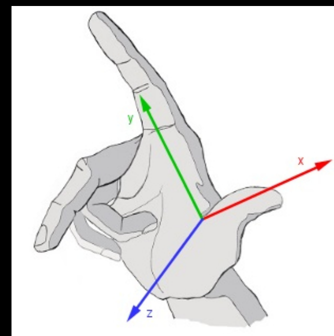


Oriented volume is positive if $(\vec{u},\vec{v},\vec{w})$ forms a right-handed co-ordinate system. Otherwise oriented volume is negative.

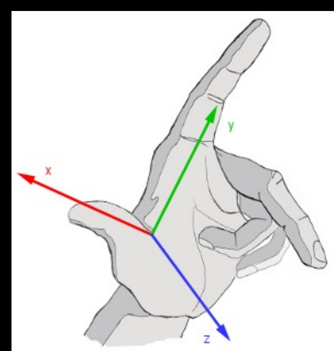
Q. How to determine $(\vec{u},\vec{v},\vec{w})$'s handedness?

Cross product and handedness of a co-ordinate system

- Right-handed co-ordinate system: $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$
  i.e., $(\hat{x} \times \hat{y}) \cdot \hat{z} > 0$ etc.
  (this is what we will use)



- Left-handed co-ordinate system: $\hat{x} \times \hat{y} = -\hat{z}$, $\hat{y} \times \hat{z} = -\hat{x}$, $\hat{z} \times \hat{x} = -\hat{y}$
  i.e., $(\hat{x} \times \hat{y}) \cdot \hat{z} < 0$ etc.



Determinants of 3×3 matrices

- Consider the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

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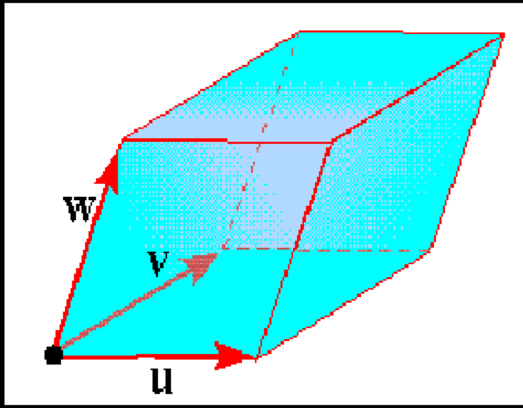
  then $\det(A)$ is the oriented volume of the parallelepiped formed by $(\vec{u}, \vec{v}, \vec{w})$



Oriented volume is positive if $(\vec{u},\vec{v},\vec{w})$ forms a right-handed co-ordinate system. Otherwise oriented volume is negative.

Q. How to determine $(\vec{u},\vec{v},\vec{w})$'s handedness?

A. Check if $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is $\gtrless 0$

Determinants of $n \times n$ matrices

- Consider the $n \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$,

  and the vectors $\vec{u}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \dots \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{23} \\ \dots \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \dots \end{bmatrix}$, $\dots$

  then $\det(A)$ is the oriented volume of the n-dimensional parallelepiped

Q. What is the determinant when, e.g., $\vec{u}_3 = \lambda \vec{u}_1 + \mu \vec{u}_2$?

Determinants of $n \times n$ matrices

- Consider the $n \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$,

  and the vectors $\vec{u}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \dots \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{23} \\ \dots \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \dots \end{bmatrix}$, $\dots$

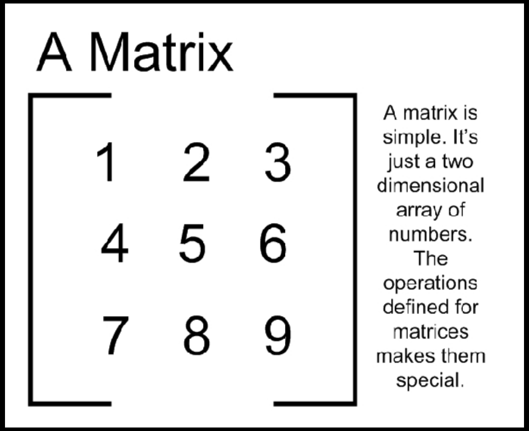
  then $\det(A)$ is the oriented volume of the n-dimensional parallelepiped

Q. What is the determinant when, e.g., $\vec{u}_3 = \lambda \vec{u}_1 + \mu \vec{u}_2$?

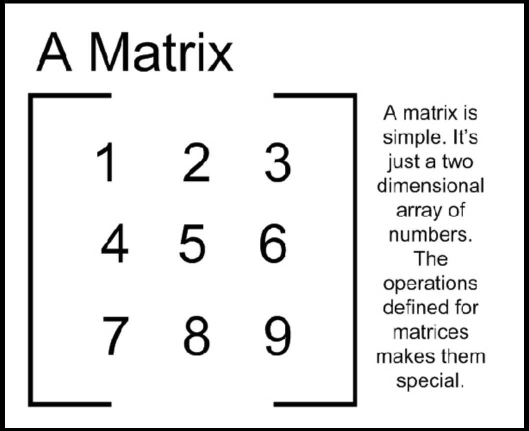
A. 0 (use the oriented volume argument!)

Introduction to transformations

Remember...


A Matrix
1 2 3
4 5 6
7 8 9
A matrix is simple. It's just a two dimensional array of numbers. The operations defined for matrices makes them special.

Remember...

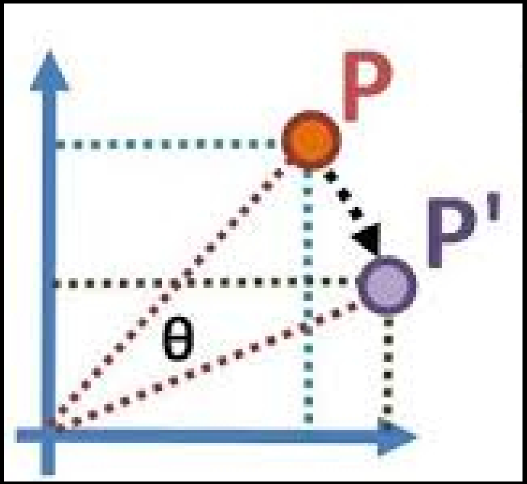


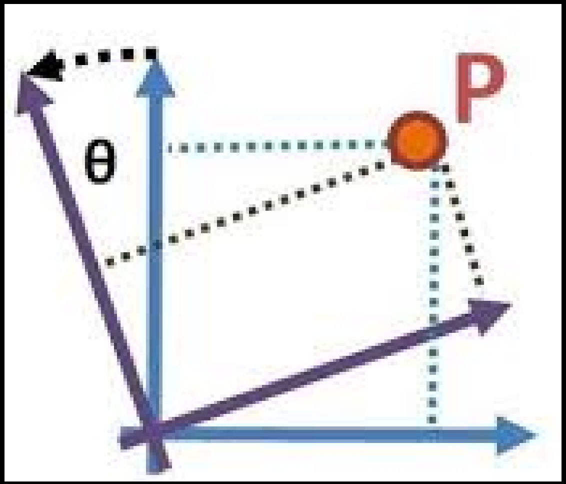
- Operations defined: they can transform vectors!
  - use them to project, scale, reflect, shear, rotate.... objects (objects means objects one point at a time)
- Point transformations: $P \to P'$ (also, active transformations)

  [point (x, y, z) represented as vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ drawn from the origin]

That brings us to... active vs. passive transformations

That brings us to... active vs. passive transformations





point transformation
(aka active transformation)

co-ordinate transformation
(aka passive transformation)

Point, or active transformations

Translation

Translation: a matrix operation?

- We translate a point P (x, y, z) by (a_x, a_y, a_z)
  i.e., $x' = x + a_x$, $y' = y + a_y$, $z' = z + a_z$

Q. How do we express this transformation as a matrix operation?
[think of using (x, y, z) as a vector $\vec{v}$ from the origin]

Translation: a matrix operation

- We translate a point P (x, y, z) by (a_x, a_y, a_z)
  i.e., $x' = x + a_x,\ y' = y + a_y,\ z' = z + a_z$

A. $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_t(\vec{a})} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix};\ \vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Translation: a matrix operation

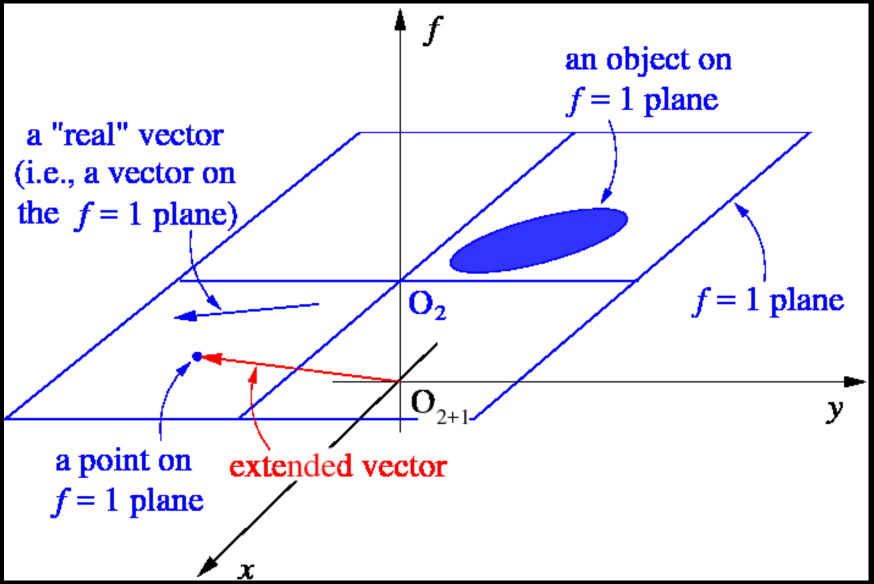
- We translate a point P (x, y, z) by (a_x, a_y, a_z)
  i.e., $x' = x + a_x$, $y' = y + a_y$, $z' = z + a_z$

A. $$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}}_{\vec{w}_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_t(\vec{a})} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\text{extended vector}} ; \vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

- From now on, will use the extended vector to reach P from the origin
  i.e., we add a fictitious dimension, meaning:

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \hat{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

How to think about an extended vector for 2D



- Note: A “real” vector $\vec{v}$, by construction, satisfies $\vec{v} \cdot \hat{f} = 0$

  e.g., the (2+1)D representation of a real vector in 2D is $\begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$

Summary

- Why? The operations defined for matrices make them special
  - matrix dimensions, special matrices (diagonal, identity, null)
- Matrix operations (addition, scalar multiplication, subtraction, matrix multiplication, transpose)
- Determinants (only for square matrices!)
- Adjoint/adjugate and inverse of matrices (only for square matrices!)
- Geometric interpretation of determinants
- Introduction to transformations
  - translation and the fictitious coordinate
- Next class: transformations (much more detailed), with matrices

Finally, references...

- Book chapter 5: Linear algebra (leave out Sec. 5.4)