Question 4: Dominating set

(total: 1.5 points)

In this question you may use the fact that the following problems are $\mathcal{N}P$-complete: PARTITION, SUBSET SUM, CLIQUE, INDEPENDENT SET,VERTEX COVER, HAMILTONIAN PATH, HAMILTONIAN CYCLE, TRAVELLING SALESMAN PROBLEM

We consider the problem DOMINATING SET. We are given a graph $G = (V, E)$. A subset D of the vertices is called a dominating set if each vertex not in D is connected to at least one vertex in D. Note that this is different from a vertex cover, which is a subset of the vertices such that each edge has one end point in the subset. In the example $\{3, 5\}$ is a dominating set, but not a vertex cover. Prove that the decision variant of DOMINATING SET is $\mathcal{N}P$-complete.

SOLUTION:

First we prove that the decision version of DOMINATINGSET is in NP.

It needs $O(n \log n)$ bits to represent the vertices and $O(n^2)$ bits to represent the edges, where n is the number of vertices. A yes instance is a subset of vertices $D \subseteq V$, which can be encoded by $O(n)$ bits and is polynomial to the input size.

For the instance, we can verify the instance by the following. We attach a check-bit for each vertices in V. It needs only $O(n)$ extra bits and is polynomial to the input size. For each $v \in D$, we mark the check-bits of v and of its neighbors as true. It takes $O(n)$ time for each vertex in D so in total the marking takes $O(n^2)$ time, which is polynomial to the input size. In the end, we check if every vertex in V is dominated by going through the check-bits and see if all of them are true. If so, the instance is a yes instance. The checking takes $O(n)$ time which is also polynomial to the input size.

Next, we prove that DOMINATINGSET is NP-hard by reduction from VERTEXCOVER.

For any instance of VERTEXCOVER, $G = (V, E)$ and integer k, we convert it to an instance of DOMINATINGSET, $G' = (V', E')$ and integer k', as the following.

- We include all the nodes in V. Furthermore, for each $(a, b) \in E$ we add a new vertex v_{ab}. More formally, $V' = V \cup V_E$, where $V_E = \{v_{ab} | (a, b) \in E\}$
- We include all the edges in E and new edges (a, v_{ab}) and (b, v_{ab}) for each new vertex v_{ab}. More formally, $E' = E \cup \{(a, v_{ab}), (b, v_{ab}) | v_{ab} \in V_E\}$
- $k' = k$

The reduction takes polynomial time with respect to the input size since there are $|E| = O(n^2)$ new vertices and $2|V_E| = O(n^2)$ new edges.

In the following we prove that there is a vertex cover with size at most k in G if and only if there is a dominating set in G' with size at most k'.

First we prove that we can construct a vertex cover C in G if there is a dominating set D in G'. Assume that all the vertices in D are in V, that is, $D \cap V_E = \phi$. In this case, for every edge $(a, b) \in E$, the corresponding vertex $v_{ab} \in V'$, $v_{ab} \notin D$ and at least one of its neighbors, a and b, is in D since D is a dominating set. We let $C = D$ and C is a vertex cover since for each edge (a, b) there is at least one endpoint in $C = D$. Hence C is a vertex cover with $|C| = |D| \leq k$.

On the other hand, for the case where $D \cap V_E \neq \phi$, we construct D' by replacing the vertices $v_{ab} \in V_E \cap D$ by one of its neighbors. That is, $D' = D \setminus \{v_{ab}\} \cup \{a\}$. The new set D' is also a dominating set since v_{ab} and the vertices which was covered by it are now dominated by a. By the replacement we eventually have a dominating set D^* such that $D^* \cup V_E = \phi$ and we can construct the instance of the vertex cover by setting $C = D^*$ where $|C| = |D^*| \leq |D| \leq k$ and the argument in the last paragraph follows.

Now we claim that if there is a vertex cover C in G, the set $D = C$ is a dominating set in G'. The reason is that for each edge $(a, b) \in E$, $a \in C$ or $b \in C$ since C is a vertex cover. Hence, for each triangle (a, b, v_{ab}) in the graph G', all the three vertices are dominated since $a \in C = D$ or $b \in C = D$. That is, $D = C$ is a vertex cover and $|D| = |C| \leq k$.

Hence, DominatingSet is NP-complete since it is in NP and it is NP-hard.