Exercises NP-completeness

Exercise 1 Knapsack problem
Consider the KNAPSACK PROBLEM. We have n items, each with weight a_j $(j = 1, ..., n)$ and value c_j $(j = 1, ..., n)$ and an integer B. All a_j and c_j are positive integers. The question is to find a subset of the items with total weight at most B and maximal value.

a) Formulate the decision problem corresponding to KNAPSACK.

b) Show that KNAPSACK belongs to NP.

c) Show that KNAPSACK is NP-complete by a reduction from SUBSET SUM.

Exercise 2. Vertex Cover
We are given an undirected graph (V, E). A vertex cover is a subset $W \subseteq V$ such that for each $(v, w) \in E$ we have $v \in W$ or $w \in W$. We consider the following problem.

> VERTEX COVER
> **Instance:** Undirected graph $G = (V, E)$, integer K.
> **Question:** Does G have a vertex cover of at most K vertices?

a) Show that VERTEX COVER belongs to the class NP.

b) Proof that the VERTEX COVER problem is NP-complete by a reduction from INDEPENDENT SET.

Exercise 3 Hamilton
The HAMILTONCYCLE problem is defined as follows: Given a **connected** graph $G = (V, E)$, does this graph contain a tour that visits each node $v \in V$ exactly once? The HAMILTONPATH problem is defined as follows: Given a **connected** graph $G = (V, E)$, does this graph contain a path that visits each node $v \in V$ exactly once? In HAMILTONPATH you can select a starting node and then visit all other nodes; you do not have to return to the starting node.

a) Prove that HAMILTONCYCLE is $\mathcal{NP}$-complete by a reduction from HAMILTONPATH.

b) Prove that HAMILTONPATH is $\mathcal{NP}$-complete by a reduction from HAMILTONCYCLE.

In exercises 4 and 5 we do not give you the problem to reduce from. However, you may use the knowledge on $\mathcal{NP}$-completeness from the lecture slides and from the previous exercises.

Exercise 4 Job scheduling
We consider the following scheduling problem. There is one machine and a set of n jobs, $J_1, \ldots, J_n$. Each job a_i ($1 \leq i \leq n$) has a processing time p_j, a profit w_j, and a deadline d_j. We must schedule the jobs on the machine, such that the machine carries out at each moment at most one job; jobs run without interruption for p_j time. Jobs that are complete before their deadline give a profit of w_j; other tasks give a profit 0. Suppose a target profit W is given. Show that the problem to decide if a schedule with profit at least W is NP-complete. **Exercise 5 Parcels and two trucks**

A company has two trucks, and must deliver a number of parcels to a number of addresses. Theywant both drivers to be home at the end of the day. This gives the following decision problem.

> **Instance:** Set V of locations, with for each pair of locations $v, w \in V$, a distance $d(v, w) \in \mathbb{N}$, a starting location $s \in V$, and an integer K.
> **Question:** Are there two cycles, that both start in s, such that every location in V is on at least one of the two cycles, and both cycles have length at most K?

Show that this problem is NP-complete.

Exercise 6 Partition
Subset Sum is defined as follows: given t nonnegative integers $b_1, \ldots, b_t$ and a nonnegative integer Q, does there exist a subset S of $\{1, 2, \ldots, n\}$ such that

$$\sum_{j \in S} b_j = Q?$$

We assume that it is given that Subset Sum is $\mathcal{NP}$-complete.

Partition is defined as follows: given n nonnegative integers $a_1, \ldots, a_n$, does there exist a subset S of $\{1, 2, \ldots, n\}$ such that

$$\sum_{j \in S} a_j = (\sum_{j \in \bar{S}} a_j)?$$

Here $\bar{S} = \{1, 2, \ldots, n\} \setminus S$.

a) Show that Partition is in $\mathcal{NP}$.

b) Prove that Partition is $\mathcal{NP}$-complete by a reduction from Subset Sum.

c) Suppose that we have a instance of Partition where the cardinality n of the set of numbers is even. Prove that Partition remains $\mathcal{NP}$-complete if we require that the subset S contains exactly $n/2$ elements.

d) * The problem Even-Odd Partition is defined as follows: given a set A of $2r$ non-negative integers' $\{a_1, \ldots, a_{2r}\}$ with $a_i \geq a_{i+1}$ $(i = 1, \ldots, 2r-1)$, does there exist a subset S of $\{1, 2, \ldots, 2r\}$ such that $\sum_{j \in S} a_j = \sum_{j \in \bar{S}} a_j$, where S contains exactly one element from $\{2i-1, 2i\}$ for every $i = 1, \ldots, r$? Prove that Even-Odd Partition is $\mathcal{NP}$-complete.

Exercise 7* **Harry Potter and the ghost**

Harry Potter tries to catch an invisible ghost. The ghost is located on a box on a 8×8 chessboard. If Harry Potter taps the box where the ghost is located with his wand, the consequences for the ghost depend on what happened before. If it is the first time a box with the ghost is tapped, the ghost will flee by moving one step in horizontal or vertical direction. However, at the second time that a box with the ghost is tapped, the ghost becomes visible and is caught. You may assume that the ghost only moves if his box is tapped.

a) Determine a strategy in which the ghost is certainly caught for which the number of required step k is as small as possible.

b) Now consider the same problem, but the ghost is located in a node of a graph (V, E). If Harry Potter taps the box where the node is located with his wand and this is the first time a node with the ghost is tapped, the ghost moves to a neighboring node. If the node were the ghost is is tapped for the second time, the ghost is caught. Again, ghost only moves if his current node is tapped. We consider the decision problem Ghost. Given a undirected graph (V, E) and an integer k, does there exist a strategy such that Harry Potter always catches the ghost in at most $n + k$ steps. Show that this decision problem is $\mathcal{NP}$-complete.