**Solutions to exercises Integer Linear Programming**

**Exercise 1**

(a) Use binary decision variables $x_{jt}$ that will model whether $X$ will teach student $j$ in period $t$ ($x_{jt} = 1$ iff this is the case). We can include the availability $a_{jt}$ in the model, but it is better to model it using preprocessing: put $x_{jt} = 0$ whenever $a_{jt} = 0$. The model then becomes

$$\max \sum_{i=1}^{n}\sum_{t=1}^{T} p_{jt}x_{jt} \quad \text{subject to}$$

$$\sum_{j=1}^{n} x_{jt} \leq 1 \quad \forall t = 1,\dots,T \quad \text{(at most 1 student per interval)}$$

$$\sum_{t=1}^{T} x_{jt} \leq q_j \quad \forall j = 1,\dots,n \quad \text{(maximum number of lessons)}$$

$$x_{jt} \in \{0,1\} \quad \forall j = 1,\dots,n; \forall t = 1,\dots,T$$

(b) Add the binary decision variables $y_j$ ($j = 1,\dots,n$) that indicate whether student $j$ will have the privilege of being taught by $X$ ($y_j = 1$ if this is the case). We again use the $x_{jt}$ variables together with the preprocessing step. The adjusted model then becomes.

$$\max \sum_{i=1}^{n}\sum_{t=1}^{T} p_{jt}x_{jt} \quad \text{subject to}$$

$$\sum_{j=1}^{n} x_{jt} \leq 1 \quad \forall t = 1,\dots,T \quad \text{(at most 1 student per interval)}$$

$$\sum_{t=1}^{T} x_{jt} = q_j y_j \quad \forall j = 1,\dots,n \quad \text{(maximum number of lessons)}$$

$$x_{jt} \in \{0,1\} \quad \forall j = 1,\dots,n; \forall t = 1,\dots,T$$

$$y_j \in \{0,1\} \quad \forall j = 1,\dots,n$$

**Exercise 2**

Use the decision variable $x_{it}$ to model the number of experienced employees recruited for project $i$ ($i = 1,\dots,n$) on day $t$ ($t = 1,\dots,T$); similarly $y_{it}$ indicates the number of unexperienced employees working on project $i$ on day $t$. The model then becomes

$$\min \sum_{i=1}^{n}\sum_{t=1}^{T} (c_1 y_{it} + c_2 x_{it}) \quad \text{subject to}$$

$$2x_{i,t} + y_{i,t} \geq w_{i,t} \quad \forall i = 1,\dots,n; \forall t = 1,\dots,T \quad \text{(workforce)}$$

$$x_{i,t} \geq u_i \qquad \forall i = 1,\dots,n; \forall t = 1,\dots,T \text{ (experience)}$$

$$\sum_{i=1}^{n} x_{i,t} \leq 25 \quad \forall t = 1,\dots,T \text{ and } \sum_{i=1}^{n} y_{i,t} \leq 55 \quad \forall t = 1,\dots,T$$

$$x_{i,t}, y_{i,t} \geq 0 \text{ and integral} \qquad \forall i = 1,\dots,n; \forall t = 1,\dots,T$$

**Exercise 3**

(a) Use a binary decision variable $x_j$ ($j = 1, \ldots, n$) to indicate whether item $j$ is included in the subset ($x_j = 1$ if this is the case). We further introduce variables $y_j$ ($j = 1, \ldots, 4$) to indicate with how much $B$ is increased using an extension of cost category $j$. The ILP model now becomes

$$\max \sum_{j=1} c_j x_j - 2y_1 - 3y_2 - 6y_3 - 10y_4 \text{ subject to}$$

$$\sum_{j=1}^{n} a_j x_j \leq B + y_1 + y_2 + y_3 + y_4$$

$$x_j \in \{0, 1\} \quad \text{for} \quad j = 1, \ldots, n,$$

$$0 \leq y_1 \leq 1; \; 0 \leq y_2 \leq 2; \; 0 \leq y_3 \leq 2; \; 0 \leq y_4 \leq 1.$$

**N.B.** Obviously, we need to be sure that you do not use the extension of type $j+1$ unless the extension of type $j$ is fully used, for $j = 1, 2, 3$. In this case, this constraint is automatically satisfied for any optimal solution because of the cost structure. Observe that we do not have enforce explicitly that the values of the variables $y_j$ are integral.

(b) Since the costs of the extensions are not increasing anymore, we must explicitly model that we use the extensions in the right order.

A possible way to acheive this is to define binary decision variables $z_i$ ($i = 1, 2, \ldots, 10$) to model of you take the $i$-th unit to extend the knapsack volume. Define $p_i$ as the cost incurred by the $i$-th unit. The model now becomes:

$$\max \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{10} p_i z_i \qquad \text{subject to}$$

$$\sum_{j=1}^{n} a_j x_j \leq B + \sum_{i=1}^{10} z_i,$$

$$z_{i+1} \leq z_i \quad \text{for} \quad i = 1, \ldots, 9,$$

$$x_j \in \{0, 1\} \quad \text{for} \quad j = 1, \ldots, n,$$

$$z_i \in \{0, 1\} \quad \text{for} \quad i = 1, \ldots, 10,$$

Observe that we now have to enforce that the $z_i$ variables are binary. Otherwise if $p_i > p_{i+1}$, then $z_i = z_{i+1} = \frac{1}{2}$ might be feasible in the LP-relaxation.

Another way of modelling is to introduce the binary decision variables $q_i$ ($i = 1, \ldots, 10$) that indicate whether you increase $B$ by exactly $i$ units (then $q_i = 1$) or not. Define $P_i = \sum_{j=1}^{i} p_j$ as the cost incurred by increasing $B$ with $i$ units ($i = 1, \ldots, 10$). The model then becomes

$$\max \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{10} P_i q_i \qquad \text{subject to}$$

$$\sum_{j=1}^{n} a_j x_j \leq B + \sum_{i=1}^{10} i q_i,$$

$$\sum_{i=1}^{10} q_i \leq 1,$$

$$x_j \in \{0, 1\} \quad \text{for} \quad j = 1, \ldots, n,$$

$$q_i \in \{0, 1\} \quad \text{for} \quad i = 1, \ldots, 10.$$

**Exercise 4.**

(a) We use the following variables:

- $y_{ij}$: number of production clusters of type $PC(i)$ $(i = 1, \ldots, 6)$ on which you produce beer type $b(j)$ $(j = 1, \ldots, 7)$.
- $y_i$: number of production clusters of type $PC(i)$ $(i = 1, \ldots, 6)$ that you lease.
- $N_j$: number of crates of beer $b(j)$ $(j = 1, \ldots, 7)$ that are sold.

The model now becomes

$$\max \quad \left\{ \sum_{j=1}^{7} p_j N_j - \sum_{i=1}^{6} c_i y_i \right\}$$

subject to

$$l(j) \leq N_j \leq u(j) \quad \forall j = 1, \ldots, 7 \quad \text{(produce enough)}$$

$$N_j \leq \sum_{i=1}^{6} a_{ij} y_{ij} \quad \forall j = 1, \ldots, 7 \quad \text{(enough PCs)}$$

$$y_i = \sum_{j=1}^{7} y_{ij} \forall i = 1, \ldots, 6; \quad (\text{\# PCs of type } i)$$

all used variables are $\geq 0$; $y_i, y_{ij}$ are integral.

(b) Let $Q_j$ be the number of crates of beer $b(j)$ $(j = 1, \ldots, 7)$ that are sold at the low price. The model now becomes:

$$\max \quad \left\{ \sum_{j=1}^{7} (p_j N_j + r_j Q_j) - \sum_{i=1}^{6} c_i y_i \right\}$$

subject to

$$l(j) \leq N_j \leq u(j) \quad \forall j = 1, \ldots, 7 \quad \text{(produce enough)}$$

$$N_j + Q_j \leq \sum_{i=1}^{6} a_{ij} y_{ij} \quad \forall j = 1, \ldots, 7 \quad \text{(enough PCs)}$$

$$y_i = \sum_{j=1}^{7} y_{ij} \forall i = 1, \ldots, 6; \quad (\text{\# PCs of type } i)$$

all used variables are $\geq 0$; $y_i, y_{ij}$ are integral.

(c) Let $z_k$ be the number of batches of raw material $k$ ($k = 1, \dots, 10$) that you buy. Now the objective function becomes:

$$\max \quad \left\{ \sum_{j=1}^{7} (p_j N_j + r_j Q_j) - \sum_{i=1}^{6} c_i y_i - \sum_{k=1}^{10} d_k z_k \right\}$$

To make sure that we buy enough raw material, we add the constraint:

$$z_k B_k \geq \sum_{j=1}^{7} (N_j + Q_j) q_{jk} \forall k = 1, \dots, 10 \quad \text{(raw material needed)}$$

Of coures $z_k$ needs to be integral.

(d) Let $W_k$ be the inventory of raw material $k$ ($k = 1, \dots, 10$) that remains. The objective function now becomes

$$\max \quad \left\{ \sum_{j=1}^{7} (p_j N_j + r_j Q_j) - \sum_{i=1}^{6} c_i y_i - \sum_{k=1}^{10} d_k z_k - \sum_{k=1}^{10} v_k W_k \right\}$$

The inventory is modelled by the constraint:

$$W_k = z_k B_k - \sum_{j=1}^{7} (N_j + Q_j) q_{jk} \forall k = 1, \dots, 10 \quad \text{(inventory } k\text{)}$$