

Introduction to Coalitional Game Theory

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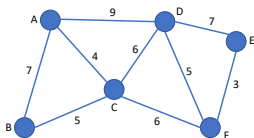
Coalitional Game Theory

- ▶ In non-coalitional games, decisions are made by individual players.
 - ▶ Binding agreements between players is not possible
 - ▶ Individual players maximise their own utilities
- ▶ In coalitional games, decisions are made by groups of players.
 - ▶ Binding agreements between players is possible
 - ▶ Individual players are utility maximisers but can benefit by cooperating

Transferable Utility Games: actions are decided by groups of players and utilities are assigned to the groups. The group utility is then distributed among individual players.

Coalitional Game Theory

- ▶ Players form groups to perform tasks
- ▶ Each group of players receives a utility (to be distributed among themselves)
- ▶ Examples:
 - ▶ Matching games based on weighted graphs



- ▶ Weighted voting game:
 - ▶ A, B, C, and D have 45, 25, 15, and 15 votes.
 - ▶ 51 votes are required to pass the \$100 million bill.

Modelling Coalitional Games

A coalitional game is $G = (N, v)$, where

- ▶ N is a set of players
- ▶ $v : 2^N \rightarrow \mathbb{R}$ is the characteristic function of the game

Example: $G = (\{1, 2\}, v)$, where

- ▶ $v(\emptyset) = 0$
- ▶ $v(\{1\}) = v(\{2\}) = 5$
- ▶ $v(\{1, 2\}) = 20$

Note :

- ▶ $v(S)$ is the value that is assigned to coalition $S \subseteq N$
- ▶ 2^N is the set of all possible coalitions
- ▶ A coalition structure CS is a partition on N
- ▶ Optimal coalition structure: $\max_{CS} \sum_{S \in CS} v(S)$

Exercise 1: Model weighted voting game as a coalitional game.

Classes of Coalitional Games

A coalitional game $G = (N, v)$ is (for any coalition $S, S' \subseteq N$):

- ▶ **Simple:** $v(S) \in \{0, 1\}$ with $v(N) = 1$ (winning/losing coalitions).
- ▶ **Additive:** $v(S \cup S') = v(S) + v(S')$ for $S \cap S' = \emptyset$.
- ▶ **Superadditive:** $v(S \cup S') \geq v(S) + v(S')$ for $S \cap S' = \emptyset$.
- ▶ **Monotonic:** $v(S) \leq v(S')$ for $S \subseteq S'$.
- ▶ **Convex:** $v(S \cup S') \geq v(S) + v(S') - v(S \cap S')$.
- ▶ **Constant-sum:** $v(S) + v(N \setminus S) = v(N)$.

Exercise 2: Show that additive implies convex implies superadditive. Show also that additive implies constant-sum, and superadditive implies monotonic.

Analysing Coalitional Games

Let $G = (N, v)$ be a coalitional game.

- ▶ v is normalised: $v(\emptyset) = 0$
- ▶ v is non-negative: $v(S) \geq 0$ for any $S \subseteq N$

An outcome $x = \langle x_1, \dots, x_k \rangle$ for a coalition S (consisting of k members in game G) is a distribution of $v(S)$ to its members such that

$$\sum_{i \in S} x_i = v(S)$$

Given the coalitional game $G = (\{1, \dots, n\}, v)$, an outcome for the grand coalition is $\langle x_1, \dots, x_n \rangle$ such that

$$\sum_{i \in N} x_i = v(N)$$

Which coalitions can be formed? \implies Which coalitions are stable?

Coalition Stability: The Core

An outcome for a coalition is **stable** if no one in that coalition can object to it.

Is the grand coalition stable?

An outcome $x = \langle x_1, \dots, x_n \rangle$ for the grand coalition is stable when each subcoalition is getting at least what it can make on its own, i.e., no one has any incentive to deviate.

The core of a coalitional game $G = (\{1, \dots, n\}, v)$ consists of all outcomes $x = \langle x_1, \dots, x_n \rangle$ for the grand coalition for which it holds:

$$\forall S \subseteq N : \sum_{i \in S} x_i \geq v(S)$$

Is the grand coalition stable? \implies Is the core non-empty?

Coalition Stability: The Core

Observation: If a vector $x = \langle x_1, \dots, x_n \rangle \in \mathbb{R}^n$ is in the core of a coalitional game $G = (\{1, \dots, n\}, v)$, then x satisfies the following constraints:

- ▶ x is **feasible**: $\sum_{i \in N} x_i \leq v(N)$
i.e., x does not allocate more than possible
- ▶ x is **efficient (pre-imputation)**: $\sum_{i \in N} x_i = v(N)$
i.e., x allocates everything
- ▶ x is **individually rational (imputation)**: $x_i \geq v(\{i\})$ for all $i \in N$
i.e., players do better in x than individually

Feasibility, efficiency, and individual rationality are necessary conditions for an outcome to be in the core, but they are not sufficient. An outcome in the core should also be *coalitional rational*.

Exercise 3: Determine the Core of the weighted voting game.

Coalitional Games with Empty Core

- ▶ Let $G = (\{1, 2, 3\}, v)$ with
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 2,$$
$$v(\{1, 2\}) = v(\{2, 3\}) = v(\{1, 3\}) = 5, \text{ and}$$
$$v(\{1, 2, 3\}) = 6.$$
- ▶ For the Core of G not to be empty, the outcomes $x = \langle x_1, x_2, x_3 \rangle$ should be:
- ▶ **feasible:** $\sum_{i \in N} x_i \leq v(N)$
$$x_1 + x_2 + x_3 \leq 6$$
- ▶ **efficient:** $\sum_{i \in N} x_i = v(N)$
$$x_1 + x_2 + x_3 = 6$$
- ▶ **individually rational:** $\forall i \in N : x_i \geq v(\{i\})$
$$x_i \geq 2$$
- ▶ **stable:** $\forall S \subseteq N : \sum_{i \in S} x_i \geq v(S)$
$$x_1 + x_2 \geq 5$$
$$x_2 + x_3 \geq 5$$
$$x_1 + x_3 \geq 5$$
- ▶ Impossible, G has an **empty** Core.

Coalitional Games with Non-Empty Core

- ▶ **Theorem:** Any convex game has a non-empty core.
- ▶ **Theorem:** Any simple game with at least one veto player has a non-empty core.

A player i in a simple game (N, v) is a veto player if for all $S \subseteq N$ we have $v(S) = 0$ if $i \notin S$.

- ▶ **Theorem:** Any coalitional game (N, v) has a non-empty core iff for all **balanced sets of weights** λ , we have:

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S)$$

A balanced set of weights λ over 2^N is a set of non-negative weights such that

$$\forall i \in N : \sum_{S: i \in S} \lambda(S) = 1$$

Stability versus Fairness

Outcomes in the Core are stable, but may not be fairly distributed.

Example: $G = (\{1, 2\}, v)$, where

- ▶ $v(\emptyset) = 0$
- ▶ $v(\{1\}) = v(\{2\}) = 5$
- ▶ $v(\{1, 2\}) = 20$

Every outcome between $(15, 5), \dots, (5, 15)$ is in the core, but some outcomes such as $15, 5$ are not fair.

Which outcomes can be considered as fair?

Marginal Contribution

The basic idea is to distribute the utility of a coalition based on the contribution of players in that coalition.

Let $G = (N, v)$ be a coalitional game. The **marginal contribution** of a player i to a coalition $S \subseteq N \setminus \{i\}$ is denoted as $\mu_i(S)$ and defined as follows:

$$\mu_i(S) = v(S \cup \{i\}) - v(S)$$

The **average marginal contribution** of a player i in a game G is defined as follows:

$$\frac{1}{2^{n-1}} \cdot \sum_{S \subseteq N \setminus \{i\}} \mu_i(S)$$

Example: $G = (\{1, 2\}, v)$, where

- ▶ $v(\emptyset) = 0$
- ▶ $v(\{1\}) = v(\{2\}) = 5$
- ▶ $v(\{1, 2\}) = 20$

Shapley Value

In some cases, the marginal contribution of player i to coalition S depends on the order in which S is formed.

Let $G = (N, v)$ be a coalitional game where $N = \{1, \dots, n\}$. The set of possible permutations of players is $\Pi(N)$. Note we have $n!$ permutations.

Let $C_i(\pi)$ denote the set of predecessors of i in the permutation $\pi \in \Pi(N)$.

The **Shapley value** of player i in the game G is defined as follows:

$$sh_i = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \mu_i(C_i(\pi))$$

Exercise 4: determine the Shapley value for all players in $G = (\{1, 2, 3\}, v)$, where

- ▶ $v(\emptyset) = 0$; $v(\{1\}) = v(\{2\}) = v(\{3\}) = 5$
- ▶ $v(\{1, 2\}) = v(\{1, 3\}) = 10$; $v(\{2, 3\}) = 20$
- ▶ $v(\{1, 2, 3\}) = 25$

Shapley Value: Properties

Let $G = (N, v)$ be a coalitional game.

- ▶ Dummy player: player i is dummy if $v(S) = v(S \cup \{i\})$ for any $S \subseteq N$
- ▶ Two players i and j are symmetric if $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq N \setminus \{i, j\}$.

Properties of the Shapley value:

1. Efficiency: $sh_1 + \dots + sh_n = v(N)$
2. Dummy: if i is a dummy player, $sh_i = 0$
3. Symmetry: if i and j are symmetric, $sh_i = sh_j$
4. Additivity: $sh_i(G_1 + G_2) = sh_i(G_1) + sh_i(G_2)$ for games $G_1 = (N, v_1)$, $G_2 = (N, v_2)$, and $G_1 + G_2 = (N, v_1 + v_2)$ defined as $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for any $S \subseteq N$.

Theorem

- ▶ *Shapley value is the only payoff distribution scheme that has properties 1- 4.*
- ▶ *In every convex game, the Shapley value outcome is in the core.*

Exercise 5: Check properties 1 - 4 for the game of previous slide.

Exercise 6: Determine the Shapley values of the players in the Weighted voting game and check if their Shapely values is in the Core of the game.