

# Introduction to Game Theory

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# Mixed Strategies

		0.2	0.8
		$c_1$	$c_2$
0.6	$r_1$	1, 1	0, 0
0.4	$r_2$	0, 0	1, 1

$\langle (0.6, 0.4), (0.2, 0.8) \rangle$

		0.2	0.5	0.3
		$c_1$	$c_2$	$c_3$
0.3	$r_1$	1, 2	2, 0	1, 1
0.7	$r_2$	2, 0	1, 2	0, 2

$\langle (0.3, 0.7), (0.2, 0.5, 0.3) \rangle$

# Mixed Strategies

		$1/2$	$1/2$
		$c_1$	$c_2$
$1/3$	$r_1$	1, 1	0, 0
$2/3$	$r_2$	0, 0	1, 1

$\langle (1/3, 2/3), (1/2, 1/2) \rangle$

		$1/3$	$1/3$	$1/3$
		$c_1$	$c_2$	$c_3$
$1/4$	$r_1$	1, 2	2, 0	1, 1
$3/4$	$r_2$	2, 0	1, 2	0, 2

$\langle (1/4, 3/4), (1/3, 1/3, 1/3) \rangle$

# Mixed Strategies and Expected Utility

**Definition:** Let  $(N, A, u)$  be a strategic game. Then:

- ▶  $\Delta(A_i)$  is the set of *mixed strategies*, i.e., set of all probability distributions over  $A_i$ .
- ▶  $\Delta(A) = \Delta(A_1) \times \cdots \times \Delta(A_n)$ , set of *mixed strategy profiles*.
- ▶ Expected utility of mixed strategy  $s \in \Delta(A)$  for player  $i$  is defined as:

$$u_i(s) = \sum_{a \in A} ( u_i(a) \cdot \prod_{j \in N} s_j(a_j) )$$

where  $a$  is a pure strategy profile,  $a_j$  is the strategy of player  $j$  in  $a$ , and  $s_j(a_j)$  is the probability value assigned to  $a_j$  by  $s_j$ .

## Notes:

- ▶ A pure strategy  $a$  is identified with the mixed strategy  $s$  for which  $s(a) = 1$ .
- ▶ Moreover,  $u_i(a)$  is interpreted as the **utility** of pure strategy  $a$  for player  $i$ , while  $u_i(s)$  is interpreted as the **expected utility** of mixed strategy  $s$  for  $i$ .

# Mixed Strategies and Expected Utility

$$u_i(s) = \sum_{a \in A} ( u_i(a) \cdot \prod_{j \in N} s_j(a_j) )$$

	0.6	0.4
0.3	1, 1	0, 0
0.7	0, 0	1, 1

$$s = \langle (0.3, 0.7), (0.6, 0.4) \rangle$$

$$\begin{aligned} u_{row}(s) &= \sum_{a \in A} ( u_{row}(a) \cdot \prod_{j \in N} s_j(a_j) ) \\ &= 1 * (0.3 * 0.6) + \\ &\quad 0 * (0.3 * 0.4) + \\ &\quad 0 * (0.7 * 0.6) + \\ &\quad 1 * (0.7 * 0.4) \\ &= 0.18 + 0.28 = 0.46 \end{aligned}$$

Exercise: Compute the expected utility of the mixed strategies on slides 2 & 3.

# The Prisoner's Dilemma

*Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at a trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done, or not to confess. If they will both do not confess, then the district attorney states he will book them on some very minor trumped up charge such as petty larceny and illegal possession of a weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped on him.*

(Luce and Raiffa, 1957, p. 95)

# The Prisoner's Dilemma

	<i>NotConfess</i>	<i>Confess</i>
<i>NotConfess</i>	2, 2	0, 3
<i>Confess</i>	3, 0	1, 1

# Pareto Efficiency

**Definition:** A **pure strategy profile**  $a \in A$  is *(weakly) Pareto efficient* if there is no pure strategy profile that is strictly better for all players, *i.e.*, if

there is no  $a' \in A$  such that for all  $i \in N$ :  $u_i(a') > u_i(a)$

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# Pareto Efficiency



**Definition:** A **pure strategy profile**  $a \in A$  is *Strongly Pareto efficient* if there is no pure strategy profile that improves some players without making others worse off, *i.e.*, if

there is no  $a' \in A$  such that for all  $i \in N$   $u_i(a') \geq u_i(a)$  and for some  $i \in N$   $u_i(a') > u_i(a)$

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# Pareto Efficiency

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*Which are the Pareto efficient strategy profiles?*

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*Which are the Pareto efficient strategy profiles?*

# Dominance

**Definition:** A **pure strategy**  $a_i$  for player  $i$  **strongly dominates** another pure strategy  $a'_i$  of  $i$  if for any strategies of the opponents,  $a_i$  is strictly better than  $a'_i$ , *i.e.*, if:

$$\text{for all } b \in A : \quad u_i(b_1, \dots, a_i, \dots, b_n) > u_i(b_1, \dots, a'_i, \dots, b_n).$$

A **pure strategy**  $a_i$  that strongly dominates all other pure strategies of player  $i$  is called a **strong dominant pure strategy** of player  $i$ .

**Definition:** A **pure strategy profile**  $a = (a_1, \dots, a_n)$  is called a **strongly dominant pure strategy equilibrium** if  $a_i$  is strongly dominant strategy for player  $i$ , for every  $i = 1, \dots, n$ .

# Dominance

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*Which are the strongly dominant strategy profiles?*

# Dominance

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*Which ones are Pareto efficient strategy profiles?*

# Dominance

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*Which are the strongly dominant strategy profiles?*

*Which ones are Pareto efficient strategy profiles?*

# Dominance

**Definition:** A **mixed strategy**  $s_i$  for player  $i$  **strongly dominates** another mixed strategy  $s'_i$  of  $i$  if for any mixed strategies of the opponents,  $s_i$  has a greater expected utility than  $s'_i$ , *i.e.*, if:

$$\text{for all } t \in \Delta(A_j) : \quad u_i(t_1, \dots, s_i, \dots, t_n) > u_i(t_1, \dots, s'_i, \dots, t_n).$$

A mixed strategy  $s_i$  of player  $i$  that strongly dominates all other mixed strategies of  $i$  is called a *strongly dominant* strategy for player  $i$ .

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# Dominance

	<i>left</i>	<i>right</i>
<i>top</i>	0, 3	3, 0
<i>middle</i>	3, 0	0, 3
<i>bottom</i>	1, 1	1, 1

# Dominance

		<i>left</i>	<i>right</i>
0.5	<i>top</i>	0, 3	3, 0
0.5	<i>middle</i>	3, 0	0, 3
0.0	<i>bottom</i>	1, 1	1, 1

Exercise: Check out other mixed strategies.

# Iterated Elimination of Dominated Strategies

Procedure of iterated elimination of dominated strategies:

- ▶ Eliminate one after another actions of player that are (weakly or strongly) dominated, until this is no longer possible
- ▶ If only one profile remains, we say the game is *dominance solvable*.

**Fact:** The strategy profiles that survive iterated elimination of weakly dominated strategies may depend on the order of elimination. This is not the case for iterated elimination of strongly dominated strategies.

# Exercise

3,1	0,0	0,0
1,1	1,2	5,0
0,1	4,0	0,0

1,1	1,1	0,0
0,0	1,2	1,2
0,2	0,0	0,3

# Best Responses

**Notation:** Given a pure (or mixed) strategy profile  $a = (a_1, \dots, a_i, \dots, a_n)$ , we use  $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$  (strategies of  $i$ 's opponent in  $a$ ), and  $(a_i, a_{-i}) = (a_1, \dots, a_i, \dots, a_n) = a$ .

**Definition:** Given  $a_{-i}$  as the pure strategies of  $i$ 's opponents, a pure strategy  $a_i$  is a **pure best response** of  $i$  to  $a_{-i}$  if for all  $b_i \in A_i$ :  $u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i})$

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**Definition:** Given  $s_{-i}$  as the mixed strategies of  $i$ 's opponents, a mixed strategy  $s_i$  is **mixed best response** of a player  $i$  to  $s_{-i}$  if for all  $t_i \in \Delta(A_i)$ :  $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$ .

# Nash Equilibrium

**Definition:** A pure strategy profile  $a$  is a **pure Nash equilibrium** if no player has an incentive to *unilaterally* deviate from  $a$ , i.e., if for all players  $i$ :

$$\text{for all } b_i \in A_i : \quad u_i(a) \geq u_i(a_1, \dots, b_i, \dots, a_n)$$

**Equivalently:** A pure strategy profile  $a$  is a **pure Nash equilibrium** if  $a_i$  is the best response to  $a_{-i}$  for all players  $i$ .

2,2	0,3
3,0	1,1

1,0	0,1
0,1	1,0

2,1	0,0
0,0	1,2

# Nash Equilibrium

**Definition:** A **mixed strategy profile**  $s$  is a *Nash equilibrium* if no player has an incentive to *unilaterally* deviate from  $s$ , i.e., if for all players  $i$ :

$$\text{for all } t_i \in \Delta(A_i) : u_i(s) \geq u_i(s_1, \dots, t_i, \dots, s_n)$$

**Equivalently:** A mixed strategy profile  $s$  is a **mixed Nash equilibrium** if  $s_i$  is the best response to  $s_{-i}$  for all players  $i$ .

2,2	0,3
3,0	1,1

1,0	0,1
0,1	1,0

2,1	0,0
0,0	1,2

# Nash's Theorem

**Theorem** (*Nash 1950*): Every strategic game with a finite number of pure strategies has a Nash equilibrium in mixed strategies.

**Remark:** *The proofs are non-constructive and use Brouwer's or Kakutani's fixed point theorems.*



# Properties of Nash Equilibrium

- ▶ Nash equilibrium is perhaps the most important solution concept for non-cooperative games, for which numerous refinements have been proposed.
- ▶ Any combination of dominant strategies is a Nash equilibrium.
- ▶ Nash equilibria are not generally Pareto efficient.
- ▶ Existence in (pure) strategies is not in general guaranteed.
- ▶ Nash equilibria are not in general unique (equilibria selection, focal points).
- ▶ Nash equilibria are not generally interchangeable.
- ▶ Payoffs in different Nash equilibria may vary.

# Finding Mixed-Strategy Nash equilibria

- ▶ Generally, it is tricky to compute mixed-strategy Nash equilibria
- ▶ But, easy if the **support** of the mixed-strategies at equilibrium can be identified

**Definition:** The *support* of a mixed strategy  $s_i$  for a player  $i$  is the set of pure strategies  $\{ a_i \mid s_i(a_i) > 0 \}$ .

## Finding Mixed-Strategy Nash equilibria

- ▶ Let the best response to  $s_{-i}$  be a mixed-strategy  $s_i$  with a support consisting of more than one action.
- ▶ **Observation:** All actions (pure strategies) in the support of strategy of  $s_i$  have the same expected utility, i.e., player  $i$  is indifferent between the actions in the support of its mixed-strategy at equilibrium.
- ▶ **Reason:** If an action  $a$  in the support of  $s_i$  has a higher expected utility than the other actions, then action  $a$  would be a better response than the mixed-strategy  $s_i$ .

2, 1	0, 0
0, 0	1, 2

**For the row player:** Suppose column player has the mixed-strategy  $(p, 1 - p)$  at equilibrium. For the row player holds that  $U_{row}(r_1) = U_{row}(r_2)$ , i.e.,

$$\begin{aligned}2 * p + 0 * (1 - p) &= 0 * p + 1 * (1 - p) \\2p &= 1 - p \\3p &= 1 \\p &= 1/3\end{aligned}$$

Exercise: Find mixed-strategy Nash equilibrium for Rock, Scissors, Paper game.

# Alternative Characterization of Nash Equilibria

**Lemma:** A mixed strategy profile  $s$  is a Nash equilibrium iff for all players  $i$

- ▶ Given  $s_{-i}$ , all actions in the support of  $s_i$  yield the same expected utility.
- ▶ Given  $s_{-i}$  no action not in the support of  $s_i$  yields a higher expected utility than any action in the support of  $s_i$ .

# Alternative Characterization of Nash Equilibria

**Lemma:** A mixed strategy profile  $s$  is a Nash equilibrium iff for all players  $i$

- ▶  $u_i(s_1, \dots, a_i, \dots, s_n) = u_i(s_1, \dots, b_i, \dots, s_n)$ , for all actions  $a_i, b_i \in A_i$  in the support of  $s_i$ .
- ▶  $u_i(s_1, \dots, a_i, \dots, s_n) \geq u_i(s_1, \dots, b_i, \dots, s_n)$ , for all actions  $a_i, b_i \in A_i$  with  $a_i$  in but  $b_i$  not in the support of  $s_i$ .

# Strictly Competitive Games (zero-sum games)

A strategic game  $G = (\{1, 2\}, A, u)$  is strictly competitive if for each strategy profile  $a$  it is the case that  $u_1(a) + u_2(a) = 0$ .

	<i>head</i>	<i>tail</i>
<i>head</i>	1, -1	-1, 1
<i>tail</i>	-1, 1	1, -1

**Lemma:** Let  $G = (\{1, 2\}, A, u)$  be a strictly competitive game. We have:

- ▶  $\max_x \min_y u_1(x, y) \leq \min_y \max_x u_1(x, y)$ .
- ▶  $\max_x \min_y u_1(x, y) = - \min_x \max_y u_2(x, y)$ .

**Exercise:** Verify the above results in the above matching Pennies game.

# Strictly Competitive Games (zero-sum games)

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	<i>head</i>	<i>tail</i>
<i>head</i>	1, -1	-1, 1
<i>tail</i>	-1, 1	1, -1

**Lemma:** Let  $G = (\{1, 2\}, A, u)$  be a strictly competitive game. We have:

- ▶ If  $(x^*, y^*) \in A$  is a Nash equilibria, then  $x^*$  is a maximinimizer for player 1 and  $y^*$  is a maximinimizer for player 2.
- ▶ If  $(x^*, y^*) \in A$  is a Nash equilibria, then  $\max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y) = u_1(x^*, y^*)$ .
- ▶ if  $\max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y) = u_1(x^*, y^*)$ ,  $x^*$  is a maximinimizer for player 1, and  $y^*$  is a maximinimizer for player 2, then  $(x^*, y^*)$  is a Nash equilibria.

**Exercise:** Design a strictly competitive game with Nash equilibria and verify the above results in that game.

# Iterated Prisoner's Dilemma

- ▶ In Prisoner's dilemma is *defect* the dominant strategy.
- ▶ Can self-interested agents cooperate? Why?
- ▶ Examples from real world: nuclear arm race, public transport
- ▶ Shadow of future: cooperation is possible because the game will be played in future again.
- ▶ Iterated Prisoner's dilemma is such a scenario.

# Axelrod's Tournament (1980)

Robert Axelrod (a political scientist) held a computer tournament designed to investigate how cooperation emerge among self interested agents.

- ▶ Computer programs play iterated prisoner's dilemma games against each other.
- ▶ Which strategy results in maximum overall payoff?
- ▶ Possible strategies followed by the submitted programs:
  - ▶ ALLD: always defect
  - ▶ ALLC: always cooperate
  - ▶ RANDOM: sometime cooperate sometimes defect
  - ▶ TIT-FOR-TAT: 1st round Cooperate. Other rounds do what the opponent did at previous round.
  - ▶ MAJORITY: 1st round cooperates. Other rounds examines the history of the opponent's actions, counting its total number of defect and cooperates. If opponent defect more often dan cooperate, then defect; otherwise cooperate.
  - ▶ JOSS: As TIT-FOR-TAT, except periodically defect.