Big Data 2018: Exercise Set 1

We hope you have watched videos 1 through 13 of the course "Introduction to Probability" as requested. Four of the exercises below were given as homework in the first lecture. Note that, next to the axioms of probability, you need results from set theory (and sometimes elementary algebra) to complete the proofs. In some cases it may be helpful to draw Venn diagrams. We use the symbols $A, B, \ldots$ rather than $e_1, e_2, \ldots$ for subsets of Ω. For easy reference, we repeat the axioms of probability here.

Probability is defined as a function from subsets of Ω to the real line $\mathbb{R}$,
that satisfies the following axioms:

1. Non-negativity: for any $A \subseteq \Omega : \mathbb{P}(A) \geq 0$.
2. $\mathbb{P}(\Omega) = 1$.
3. Additivity: If $A \cap B = \emptyset$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Where indicated, exercises have been taken from the following book:

Hodges and Lehmann: J.L. Hodges, Jr. and E.L. Lehmann, Basic Concepts of Probability and Statistics (Second Edition), Holden-Day, 1970.

Exercises

1. Prove that $\mathbb{P}(\emptyset) = 0$.
2. Prove that if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
3. Prove that
$$\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A).$$
Here $\bar{A}$ denotes the complement of A (with respect to Ω).
4. Prove that $\mathbb{P}(A) \leq 1$ for every $A \subseteq \Omega$.
5. Prove that
$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B}).$$
6. Prove that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

7. Prove that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$. This is called the union bound.

8. We say A and B are independent events if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Show that if A and B are independent events, then A and $\bar{B}$ are also independent.
   Hint: use the answers to exercise 3 and 5.

9. (after Hodges and Lehmann) Five wine experts each are asked to compare three glasses of wine, labeled as A, B, and C. Suppose that the three glasses are in fact indistinguishable (they are poured from the same bottle) so that each of the experts without any relation to the decision of the other experts is equally likely to rank the wines in any of the six possible orders. What is the probability that all judges

   (a) assign the ranking ABC ?

   (b) rank A highest ?

   (c) prefer A to B ?

10. (after Hodges and Lehmann) In a study of the performance in course A and B, a department finds that the following model suitably describes the distribution of grades (measured on a 5-point scale) for students completing both courses:

A \ B	0	1	2	3	4
0	.00	.01	.00	.00	.00
1	.03	.05	.04	.00	.00
2	.01	.04	.26	.05	.00
3	.00	.03	.11	.15	.03
4	.00	.00	.03	.07	.09

   Find the probability that a student will

   (a) do better in course B than in course A,

   (b) get the same grade in both courses,

   (c) change his grade by more than one grade point from course A to course B,

   (d) get a total of at least six grade points,

   (e) get a total of exactly six grade points.

11. (Hodges and Lehmann) Assuming that in three-child families the 8 cases MMM, MMF, ... , FFF are equally likely, find the probabilities of the following events:

   (a) at least one boy
   (b) at least two boys
   (c) exactly one boy
   (d) exactly two boys
   (e) at most one boy
   (f) more boys than girls
   (g) at least one girl and one boy
   (h) the oldest a boy
   (i) oldest a boy and youngest a girl
   (j) no girl younger than a boy