Big Data 2018: Solutions Set 1

1. To prove that $\mathbb{P}(\emptyset) = 0$, first note that $A \cap \emptyset = \emptyset$ for any $A \subseteq \Omega$, so that we can write (axiom 3) $\mathbb{P}(A \cup \emptyset) = \mathbb{P}(A) + \mathbb{P}(\emptyset)$. Furthermore, $A \cup \emptyset = A$, and therefore we have $\mathbb{P}(A) = \mathbb{P}(A) + \mathbb{P}(\emptyset)$. It follows that $\mathbb{P}(\emptyset) = 0$.

2. To prove an implication, we may assume the antecedent and show that the consequent follows.

   (a) $A \subseteq B$ (assumption)

   (b) $B = A \cup (B \cap \bar{A})$ ((a), set theory)

   (c) $A \cap (B \cap \bar{A}) = \emptyset$ (set theory)

   (d) $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap \bar{A})$ ((b), (c), axiom 3)

   (e) $\mathbb{P}(B \cap \bar{A}) \geq 0$ (axiom 1)

   (f) $\mathbb{P}(A) \leq \mathbb{P}(B)$ ((d),(e), algebra)

3. By set theory, we have $A \cup \bar{A} = \Omega$, and $A \cap \bar{A} = \emptyset$. So

$$\mathbb{P}(\Omega) = \mathbb{P}(A \cup \bar{A}) = \mathbb{P}(A) + \mathbb{P}(\bar{A}) = 1.$$

   Hence the desired result follows.

4. This follows from the previous result and axiom 1.

5. By set theory, we have $A = (A \cap B) \cup (A \cap \bar{B})$. Since $(A \cap B) \cap (A \cap \bar{B}) = \emptyset$ we have the desired result by additivity (axiom 3).

6. (a) $A \cup B = (A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (A \cap B)$ (set theory)

   (b) $A = (A \cap \bar{B}) \cup (A \cap B)$ (set theory)

   (c) $B = (\bar{A} \cap B) \cup (A \cap B)$ (set theory)

   (d) $\mathbb{P}(A \cup B) = \mathbb{P}(A \cap \bar{B}) + \mathbb{P}(\bar{A} \cap B) + \mathbb{P}(A \cap B)$ ((a), axiom 3)

   (e) $\mathbb{P}(A) = \mathbb{P}(A \cap \bar{B}) + \mathbb{P}(A \cap B)$ ((b), axiom 3)

   (f) $\mathbb{P}(B) = \mathbb{P}(\bar{A} \cap B) + \mathbb{P}(A \cap B)$ ((c), axiom 3)

   The rest of the proof is elementary algebra.

7. This follows from the previous result and axiom 1.

8.

$$
\begin{aligned}
\mathbb{P}(A \cap \bar{B}) &= \mathbb{P}(A) - \mathbb{P}(A \cap B) && \text{(exercise 5)} \\
&= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) && \text{(independence of } A \text{ and } B\text{)} \\
&= \mathbb{P}(A)(1 - \mathbb{P}(B)) && \text{(elementary algebra)} \\
&= \mathbb{P}(A)\mathbb{P}(\bar{B}) && \text{(exercise 3)}
\end{aligned}
$$

9. (after Hodges and Lehmann)

   (a) For each judge, the probability that the judge assigns the ranking ABC is $\frac{1}{6}$. Since all five judges decide independently, the joint probability is $(\frac{1}{6})^5 = \frac{1}{7776}$.

   (b) $(\frac{1}{3})^5 = \frac{1}{243}$

   (c) $(\frac{1}{2})^5 = \frac{1}{32}$

10. (after Hodges and Lehmann)

   (a) The relevant part of the outcome space is made gray. This adds up to 0.13.

A \ B	0	1	2	3	4
0	.00	.01	.00	.00	.00
1	.03	.05	.04	.00	.00
2	.01	.04	.26	.05	.00
3	.00	.03	.11	.15	.03
4	.00	.00	.03	.07	.09

   (b) This time it adds up to 0.55.

A \ B	0	1	2	3	4
0	.00	.01	.00	.00	.00
1	.03	.05	.04	.00	.00
2	.01	.04	.26	.05	.00
3	.00	.03	.11	.15	.03
4	.00	.00	.03	.07	.09

   (c) This adds up to 0.07.

A \ B	0	1	2	3	4
0	.00	.01	.00	.00	.00
1	.03	.05	.04	.00	.00
2	.01	.04	.26	.05	.00
3	.00	.03	.11	.15	.03
4	.00	.00	.03	.07	.09

(d) This adds up to 0.37.

A \ B	0	1	2	3	4
0	.00	.01	.00	.00	.00
1	.03	.05	.04	.00	.00
2	.01	.04	.26	.05	.00
3	.00	.03	.11	.15	.03
4	.00	.00	.03	.07	.09

(e) This adds up to 0.18.

A \ B	0	1	2	3	4
0	.00	.01	.00	.00	.00
1	.03	.05	.04	.00	.00
2	.01	.04	.26	.05	.00
3	.00	.03	.11	.15	.03
4	.00	.00	.03	.07	.09

11. (Hodges and Lehmann) (a) 7/8 (b) 4/8 (c) 3/8 (d) 3/8 (e) 4/8 (f) 4/8 (g) 6/8 (h) 4/8 (i) 2/8 (j) 4/8