Frequent Itemset Mining

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Battling Size

The previous time we saw that Big Data

- has a size problem

And we ended by noting that

- sampling may help us to battle that problem

From now on, our one goal is

- to show how sampling helps in

  Frequent Itsemset Mining

Why this context?

- Frequent pattern mining is an important topic in unsupervised data mining with many applications
- Moreover, there are good theoretical results
  - and the theory is based on sampling for classification
  - which is a basic problem in machine learning

Posh Food Ltd

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You own an upmarket supermarket chain, selling mindbogglingly expensive food and drinks to customers with more money than sense

- and you wonder how you can wring even more money out of your customers

You think it might be a good idea to suggest items that go well with what they already plan to buy.

- e.g., that a bottle of Krug Clos d'Ambonnay goes very well with the Iranian Almas Beluga Caviar they just selected.

But, unfortunately, you are not as rich as your clientèle, so you actually don't know

- so, you decide to look for patterns in your data
- sets of items – also known as itemsets – that your customers regularly buy together.

You decide to mine your data for all frequent itemsets

- all sets of items that have been bought more then θ times in your chain.

Your First Idea

You collect all transactions over the past year in your chain

- there turn out to be millions of them, a fact that makes you happy.

Since you only sell expensive stuff

- nothing below a thousand or so; you do sell potatoes, but only "La Bonnotte"

you only have a few thousand different items for sale.

Since, you want to know which sets of items sell well,

- you decide to list simply all sets of items
- and check whether or not they were sold together θ times or more.

And this is a terrible idea!

- as you discover when you break off the computation after a week long run

Why is it Naive?

A set with n elements has 2^n subsets

- $2^{1000} \approx 10^{301}$

The universe is about 14×10^9 years old

- and a year has about 31 million seconds

A modern CPU runs at about 5Ghz $= 5 \times 10^9$ Hz

- which means that the universe is about $14 \times 10^9 \times 31 \times 10^6 \times 5 \times 10^9 = 2,2 \times 10^{25}$ clockticks old

So, if your database fits into the CPU cache

- and you can check one itemset per clocktick
- and you can parallelise the computation perfectly

You would need

- $10^{301}/(2,2 \times 10^{25}) \approx 5 \times 10^{275}$ computers that have been running in parallel since the big bang to finish about now!

The number of elementary particles in the observable universe is, according to Wikipedia, about 10^{97}

- excluding dark matter

A New Idea

Feeling disappointed, you idly query your database.

- how many customers bought your favourite combination?
- Wagyu beef with that beautiful white Italian truffle accompanied by a bottle of Romanée-Conti Grand Cru

And to your surprise you find 0! You search for a reason

- plenty people buy Wagyu or white truffle or Romanée-Conti – actually, they belong to your top sold items
- quite a few people buy Wagyu and Romanée-Conti and the same holds for Wagyu and white truffle
- but no-one buys white truffle and Romanée-Conti
- those Philistines prefer a Chateau Pétrus with their truffle!
  - on second thoughts: not a bad idea

Clearly you cannot buy Wagyu and white truffle and Romanée-Conti more often

- than you buy white truffle and Romanée-Conti!

A Priori

With this idea in mind, you implement the A Priori algorithm

- simple levelwise search
- you only check sets of n elements for which all subsets of $n-1$ elements are frequent

After you finished your implementation

- you throw your data at it
- and a minute or so later you have all your frequent itemsets.

In principle, all subsets could be frequent

- and A Priori would be as useless as the naive idea

But, fortunately, people do not buy that many different items in one transaction

- whatever you seem to observe on your weekly shopping run

Transaction Databases

After this informal introduction, it is time to become more formal

- we have a set of *items* $\mathcal{I} = \{i_1, \ldots, i_n\}$
  - representing, e.g., the items for sale in your store
- a *transaction* t is simply a subset of $\mathcal{I}$, i.e., $t \subseteq \mathcal{I}$
  - or, more precisely, a pair (tid, t) in which $tid \in \mathbb{N}$ is the (unique) tuple id and t is a transaction in the sense above
- Note that there is *no* count of how many copies of i_j the customer bought, just a record of the fact whether or not you bought i_j
  - you can easily represent that in the same scheme if you want
- A database D is a set of transactions
  - all with a unique tid, of course
  - if you don't want to bother with tid's, D is simply a bag of transactions

Frequent Itemsets

Let D be a transaction database over $\mathcal{I}$

- an *itemset* I is a set of items (duh), $I \subseteq \mathcal{I}$
- itemset I *occurs* in transaction (tid, t) if $I \subseteq t$
- the *support* of an itemset in D is the number of transaction it occurs in

$$supp_D(I) = |\{(tid, t) \in D \mid I \subseteq t\}|$$

- note that sometimes the *relative* form of support is used, i.e.,

$$supp_D(I) = \frac{|\{(tid, t) \in D \mid I \subseteq t\}|}{|D|}$$

- An itemset I is called *frequent* if its support is equal or larger than some user defined minimal threshold θ

$$I \text{ is frequent in } D \Leftrightarrow supp_D(I) \geq \theta$$

Frequent Itemset Mining

The problem of frequent itemset mining is given by

> *Given a transaction database D over a set of items $\mathcal{I}$, find all itemsets that are frequent in D given the minimal support threshold θ.*

The original motivation for frequent itemset mining comes from *association rule* mining

- an association rule is given by a pair of disjoint itemsets X and Y ($X \cap Y = \emptyset$), it is denoted by

$$X \rightarrow Y$$

- where $P(XY) \geq \theta_1$, is the (relative) support of the rule
  - i.e., the relative $supp_D(X \cup Y) = supp_D(XY) \geq \theta_1$
- and $P(Y|X) \geq \theta_2$ is the confidence of the rule
  - i.e., $\frac{supp_D(XY)}{supp_D(X)} \geq \theta_2$

Association Rules

The motivation of association rule mining is simply the observation that

- people that buy X also tend to buy Y
    - for suitable thresholds θ_1, θ_2
- which may be valuable information for sales and discounts

But then you might think

- *correlation is no causation*
- all you see is correlation

And you are completely right

- but why would the supermarket manager care?
- if he sees that ice cream and swimming gear are positively correlated
- he knows that if sales of the one goes up, so will (likely) the sales of the other
- whether or not there is a causal relation or both are caused by an external factor like nice weather.

Discovering Association Rules

Given that there are two thresholds, discovering association rules is usually a two step procedure

- first discover all frequent itemsets wrt θ_1
- for each such frequent itemset I consider all partitions of I to check whether or not that partition satisfies the second condition
  - actually one should be a bit careful so that you don't consider partitions that cannot satisfy the second requirement
  - which is very similar to the considerations in discovering the frequent itemsets

The upshot is that the difficult part is

- discovering the frequent itemsets

Hence, most of the algorithmic effort has been put

- in exactly that task

Later on it transpired that frequent itemsets

- or, more general, frequent patterns

have a more general use, we will come back to that, briefly, later

Discovering Frequent Itemsets

Obviously, simply checking all possible itemsets to see whether or not they are frequent is not doable

- $2^{|\mathcal{I}|} - 1$ is rather big, even for small stores

Fortunately, there is the A Priori property

$$I_1 \subseteq I_2 \Rightarrow supp_D(I_1) \geq supp_D(I_2)$$

Proof

$$\begin{aligned}\{(tid, t) \in D \mid I_2 \subseteq t\} &= \{(tid, t) \in D \mid I_1 \subseteq I_2 \subseteq t\} \\ &\subseteq \{(tid, t) \in D \mid I_1 \subseteq t\}\end{aligned}$$

since $I_1 \subseteq I_2 \subseteq t$ is a stronger requirement than $I_1 \subseteq t$. So, we have

$$\begin{aligned}supp_D(I_2) &= |\{(tid, t) \in D \mid I_2 \subseteq t\}| \\ &\leq |\{(tid, t) \in D \mid I_1 \subseteq t\}| = supp_D(I_1)\end{aligned}$$

If I_1 is *not* frequent in D, neither is I_2

Levelwise Search

Hence, we know that:

if $Y \subseteq X$ and $supp_D(X) \geq t_1$, then $supp_D(Y) \geq t_1$.
and conversely,
if $Y \subseteq X$ and $supp_D(Y) < t_1$, then $supp_D(X) < t_1$.

In other words, we can search *levelwise* for the frequent sets. The level is the number of items in the set:

> *A set X is a* candidate frequent set *iff all its subsets are frequent.*

Denote by $C(k)$ the sets of k items that are potentially frequent (the candidate sets) and by $F(k)$ the frequent sets of k items.

Apriori Pseudocode

Algorithm 1 Apriori(θ, $\mathcal{I}$, D)

```
1:  $C(1) \leftarrow \mathcal{I}$
2:  $k \leftarrow 1$
3:  while $C(k) \neq \emptyset$ do
4:      $F(k) \leftarrow \emptyset$
5:      for all $X \in C(k)$ do
6:          if $supp_D(X) \geq \theta$ then
7:              $F(k) \leftarrow F(k) \cup \{X\}$
8:          end if
9:      end for
10:     $C(k+1) \leftarrow \emptyset$
11:     for all $X \in F(k)$ do
12:         for all $Y \in F(k)$ that share $k-1$ items with $X$ do
13:             if All $Z \subset X \cup Y$ of $k$ items are frequent then
14:                 $C(k+1) \leftarrow C(k+1) \cup \{X \cup Y\}$
15:             end if
16:         end for
17:     end for
18:     $k \leftarrow k+1$
19: end while
```

Example: the data

tid	Items
1	ABE
2	BD
3	BC
4	ABD
5	AC
6	BC
7	AC
8	ABCE
9	ABC

Minimum support $= 2$

Example: Level 1

tid	Items
1	ABE
2	BD
3	BC
4	ABD
5	AC
6	BC
7	AC
8	ABCE
9	ABC

Candidate	Support	Frequent?
A	6	Yes
B	7	Yes
C	6	Yes
D	2	Yes
E	2	Yes

Example: Level 2

tid	Items
1	ABE
2	BD
3	BC
4	ABD
5	AC
6	BC
7	AC
8	ABCE
9	ABC

Candidate	Support	Frequent?
AB	4	Yes
AC	4	Yes
AD	1	No
AE	2	Yes
BC	4	Yes
BD	2	Yes
BE	2	Yes
CD	0	No
CE	1	No
DE	0	No

Example: Level 3

tid	Items
1	ABE
2	BD
3	BC
4	ABD
5	AC
6	BC
7	AC
8	ABCE
9	ABC

Candidate	Support	Frequent?
ABC	2	Yes
ABE	2	Yes

Level 3: For example, ABD and BCD are not level 3 candidates.
Level 4: There are no level 4 candidates.

Order, order

Lines 10-11 of the algorithm leads to multiple generations of the set $X \cup Y$.

For example, the candidate ABC is generated 3 times

1. by combining AB with AC
2. by combining AB with BC
3. by combining AC with BC

Order, order

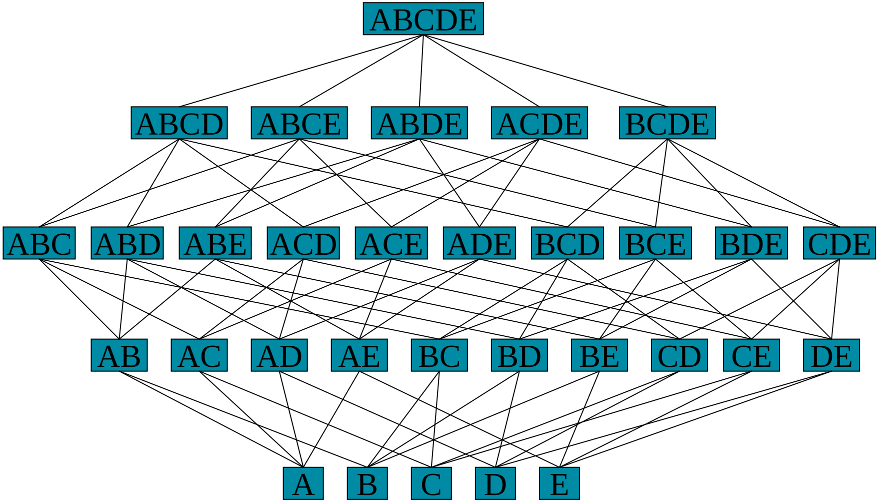
The solution is to place an order on the items.

```
for all $X \in F(k)$ do
  for all $Y \in F(k)$ that share the first $k-1$ items with $X$ do
    if All $Z \subset X \cup Y$ of $k$ items are frequent then
      $C(k+1) \leftarrow C(k+1) \cup \{X \cup Y\}$
    end if
  end for
end for
```

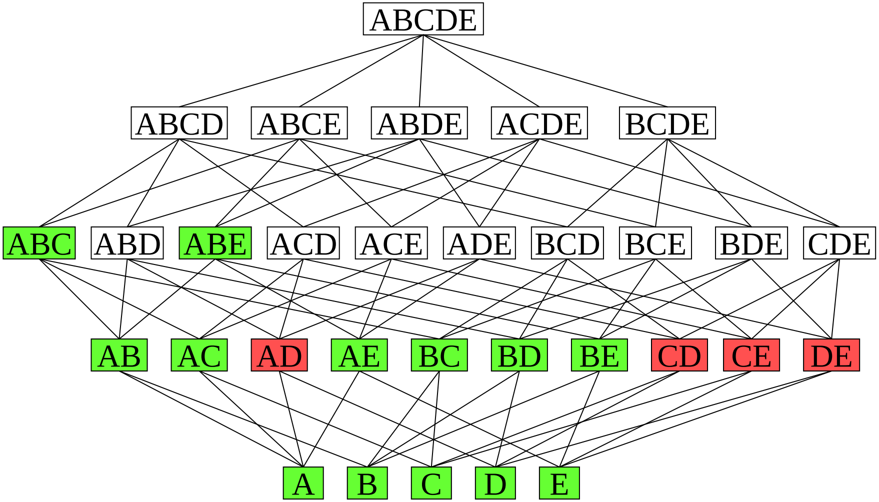
Now the candidate ABC is generated just once, by combining AB with AC.

The order itself is arbitrary, as long as it is applied consistently.

The search space



Item sets counted by Apriori



The Complexity of Apriori

Take a database with just 1 tuple consisting completely of 1's and set minimum support to 1. Then, all subsets of $\mathcal{I}$ are frequent! Hence, the worst case complexity of level wise search is $O(2^{|\mathcal{I}|})$!

However, suppose that D is *sparse* (by far the most values are 0), then we *expect* that the frequent sets have a maximal size m with $m << |\mathcal{I}|$

If that expectation is met, we have a worst case complexity of:

$$O\left(\sum_{j=1}^{m} \binom{|\mathcal{I}|}{j}\right) = O(|\mathcal{I}|^m) << O(2^{|\mathcal{I}|})$$

More General

Apriori is not only a good idea for itemset mining

- it is applicable in pattern mining in general
- provided that some simple conditions are met

To explain this more general setting

- we briefly discuss partial orders
- lattices and
- Galois connections

Partial Orders

A partially ordered set $(X, \preceq)$ consists of

- a set X
- and a partial order $\preceq$ on X
- that is, $\forall x, y, z \in X$:
  1. $x \preceq x$
  2. $x \preceq y \wedge y \preceq x \Rightarrow x = y$
  3. $x \preceq y \wedge y \preceq z \Rightarrow x \preceq z$

An element $x \in X$ is an upperbound of a set $S \subseteq X$ if

$$\forall s \in S : s \preceq x$$

It is the least upperbound, aka join, of S if

$$\forall y \in \{y \in X \mid \forall s \in S : s \preceq y\} : x \preceq y$$

Lowerbounds and greatest lowerbounds, aka meet, are defined dually

Lattices

A partially ordered set $(X, \preceq)$ is a lattice if each two elements subset $\{x, y\} \subseteq X$

- has a join, denoted by $x \vee y$
- and a meet, denoted by $x \wedge y$

If for $S, T \subseteq X$, $\bigvee S, \bigvee T, \bigwedge S$, and $\bigwedge T$ exist, then

- $\bigvee (S \cup T) = (\bigvee S) \vee (\bigvee T)$
- $\bigwedge (S \cup T) = (\bigwedge S) \wedge (\bigwedge T)$

A lattice is bounded if it has a largest element 1, sometimes denoted by $\top$, and a smallest element 0, sometimes denoted by $\bot$:

- $\forall x \in X : 0 \preceq x \preceq 1$

A lattice is complete

- if all its subsets have a join and a meet

Note that it immediately follows that

- each complete lattice is bounded
- each finite lattice is complete

Properties of Lattices

It is easy to see that for any lattice we have that $\vee$ and $\wedge$ are

- idempotent $x \vee x = x \wedge x = x$
- commutative $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$
- associative $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Moreover, they obey the absorption laws:

- $x \vee (x \wedge y) = x$
- $x \wedge (x \vee y) = x$

Note that idempotency of $\vee$ and $\wedge$ are a direct consequence of the absorption laws

- in fact, they are a special case

Rather than starting from a partial order $\preceq$ one can define lattices algebraically

- with two operators $\vee$ and $\wedge$
- that follow the commutative, associative and absorption laws given above

Two Examples of Lattices

In our discussion of frequent itemset mining we have already met two lattices

1. The itemsets, $(P(\mathcal{I}), \subseteq)$
   - where $\cup$ is the join $\vee$
   - $\cap$ is the meet $\wedge$
   - and the smallest and largest elements are $\emptyset$ and $\mathcal{I}$, respectively
2. Subsets of the database $(P(D), \subseteq)$
   - with the same operators as above
   - and $\emptyset$ and D as minimal and maximal element

Both are finite and complete and we know that they have distributive properties:

- $x \vee (y \wedge z) = x \cup (y \cap z) = (x \cup y) \cap (x \cup z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (a \wedge y) \vee (x \wedge z)$

Both are the nicest type of lattice you can imagine

- as is any subset lattice

Galois Connections

Let $(A, \leq)$ and $(B, \preceq)$ be two partially ordered sets. and let $F : A \to B$ and $G : B \to A$ be two functions

- (F, G) is a monotone Galois connection iff

$$\forall a \in A, b \in B : F(a) \preceq b \Leftrightarrow a \leq G(b)$$

- (F, G) is a anti-monotone (antitone) Galois connection iff

$$\forall a \in A, b \in B : b \preceq F(a) \Leftrightarrow a \leq G(b)$$

In the monotone case we have for the *closure* operators $GF : A \to A$ and $FG : B \to B$ that

- $a \leq GF(a)$ and $FG(b) \preceq b$

While in the anti-monotone case we have for these closure operators that

- $a \leq GF(a)$ and $b \preceq FG(b)$

A Galois Connection

There is an easy Galois connection between the two lattices $(P(\mathcal{I}), \subseteq)$ and $(P(D), \subseteq)$:

- define $F : P(\mathcal{I}) \to P(D)$ by

$$F(I) = \{t \in D \mid I \subseteq t\}$$

- define $G : P(D) \to P(\mathcal{I})$ by

$$\begin{aligned} G(E) &= \{i \in I \mid \forall t \in E : i \in t\} \\ &= \bigcap_{t \in E} t \end{aligned}$$

Now, note that for $I \in P(\mathcal{I})$ and $E' \in P(D)$ we have that

$$[E' \subseteq F(I)] \Leftrightarrow [\forall t \in E' : I \subseteq t] \Leftrightarrow \left[I \subseteq \bigcap_{t \in E'} t\right] \Leftrightarrow [I \subseteq G(E')]$$

That is, the connection is anti-monotone

Closed Itemsets

With these F and G, we have the mapping

$$GF : P(\mathcal{I}) \to P(\mathcal{I})$$

If an itemset $I \in P(\mathcal{I})$ is a fixed point of GF

$$GF(I) = I$$

then I is called a *closed* itemset.

It is easy to see that $I \in P(\mathcal{I})$ is closed iff

- $\forall i \in \mathcal{I} : i \notin I \to supp_D(I \cup \{i\}) < supp_D(I)$

Call an itemset $J \in P(\mathcal{I})$ maximal iff

- J is frequent in D
- $\forall K \in P(\mathcal{I}) : J \subset K \to K$ is *not* frequent

Then we have

- maximal itemsets are closed

A Condensed Representation

Let C be the set of all closed frequent item sets and let $J \in P(\mathcal{I})$.

- if $\forall I \in C : J \not\subset I$ then J is not frequent
  - there is a maximal $K \in C$ such that $K \subset J$ and thus J is not frequent
- if $\exists I \in C : J \subset I$, then J is frequent and we know its frequency
  - just look at the frequency of all $I \in C : J \subset I$ and take the .... frequency of those. Since that that itemset is frequent, so is J.

In other words C tells you all there is to know about the set of frequent itemsets.

- it is a condensed representation of the set of all frequent itemsets

The Power of Anti-Monotone

The reason that the A Priori principle holds

- and thus that the Apriori algorithm works

is that the Galois connection between $P(\mathcal{I})$ and $P(D)$ is anti-monotone, because that means that

- $I_1 \subseteq I_2 \Rightarrow F(I_1) \supseteq F(I_2)$
- and $supp_D(I) = |F(I)|$

In other words, we can use the Apriori Algorithm on *any* anti-monotone Galois Connection.

We'll explain this in more detail on the following few slides following Manilla and Toivonen, Levelwise Search and Borders of Theories in Knowledge Discovery, DMKD, 1997.

Theory Mining

Given a language $\mathcal{L}$

- for defining subgroups of the database
  - one example is $\mathcal{L} = P(\mathcal{I})$

and a predicate q that

- determines whether or not $\phi \in \mathcal{L}$ describes an interesting subset of D
- i.e., whether or not $q(\phi, D)$ is true or not
  - an example of q is $supp_D(I) \geq \theta$

The task is to compute the theory of D with respect to $\mathcal{L}$ and q. That is, to compute

$$\mathcal{T}\langle(\mathcal{L}, D, q) = \{\phi \in \mathcal{L} \mid q(\phi, D)\}$$

Now, if $\mathcal{L}$ is a finite set with a partial order $\preceq$ such that

$$\psi \preceq \phi \Rightarrow [q(\phi, D) \rightarrow q(\psi, D)]$$

we have the anti-monotonicity to use Apriori

Queries and Consequences

Since $\mathcal{L}$ defines subgroups of the database

- it is essentially a query language

Most query languages naturally have a partial order

- either "syntactically" $D \vdash \phi \to D \vdash \psi$
- or semantically $D \models \phi \to D \models \psi$
- or both can be used (think of monomials)

Furthermore, note that query languages can be defined for many different types of data, e.g.,

- graphs
- data streams
- text

For all these types, and many more, we can define pattern languages

- and compute all frequent patterns using levelwise search.

Complexity: the database perspective

We only looked at the complexity wrt the number of items of our table. But, that is not the only aspect: what about the role of the database?

- If we check each itemset separately, we need as many passes over the database as there are candidate frequent sets.
- If at each level we first generate all candidates and check all of them in one pass, we need as many passes as the size of the largest candidate set.

If the database does not fit in main memory, such passes are costly in terms of I/O.

Can we use sampling? Of course we can!

More on this next time