# Log-Linear Expansion: Example

The probability function of bivariate Bernoulli random vector $(X_1, X_2)$ is determined by

$$P(x_1, x_2) = p(x_1, x_2)$$

where $p(x_1, x_2)$ is the table of probabilities:

| $p(x_1, x_2)$ | $x_2 = 0$ | $x_2 = 1$ | Total |
|---|---|---|---|
| $x_1 = 0$ | $p(0, 0)$ | $p(0, 1)$ | $p_1(0)$ |
| $x_1 = 1$ | $p(1, 0)$ | $p(1, 1)$ | $p_1(1)$ |
| Total | $p_2(0)$ | $p_2(1)$ | 1 |

We can write this as a single formula:

$$P(x_1, x_2) = p(0, 0)^{(1-x_1)(1-x_2)} p(0, 1)^{(1-x_1)x_2} p(1, 0)^{x_1(1-x_2)} p(1, 1)^{x_1 x_2}$$

Taking logarithms and collecting terms in $x_1$, $x_2$, and $x_1 x_2$ gives:

$$\begin{aligned}
\log P(x_1, x_2) &= (1 - x_1)(1 - x_2) \log p(0, 0) + (1 - x_1)x_2 \log p(0, 1) + x_1(1 - x_2) \log p(1, 0) + x_1 x_2 \log p(1, 1) \\
&= \log p(0, 0) - x_2 \log p(0, 0) - x_1 \log p(0, 0) + x_1 x_2 \log p(0, 0) + x_2 \log p(0, 1) - x_1 x_2 \log p(0, 1) + \\
&+ x_1 \log p(1, 0) - x_1 x_2 \log p(1, 0) + x_1 x_2 \log p(1, 1) \\
&= \log p(0, 0) + \log \frac{p(1, 0)}{p(0, 0)}\, x_1 + \log \frac{p(0, 1)}{p(0, 0)}\, x_2 + \log \frac{p(1, 1)p(0, 0)}{p(0, 1)p(1, 0)}\, x_1 x_2 \\
&= u_0 + u_1 x_1 + u_2 x_2 + u_{12} x_1 x_2,
\end{aligned}$$

where

$$\begin{aligned}
u_0 &= \log p(0, 0) \\
u_1 &= \log \frac{p(1, 0)}{p(0, 0)} \\
u_2 &= \log \frac{p(0, 1)}{p(0, 0)} \\
u_{12} &= \log \frac{p(1, 1)p(0, 0)}{p(0, 1)p(1, 0)}
\end{aligned}$$

Elementary properties of logarithms:

1. $\log a^b = b \log a$,
2. $\log \frac{a}{b} = \log a - \log b$, and
3. $\log ab = \log a + \log b$.