Measures and metrics intro

Scientific Perspectives on GMT 2019/2020

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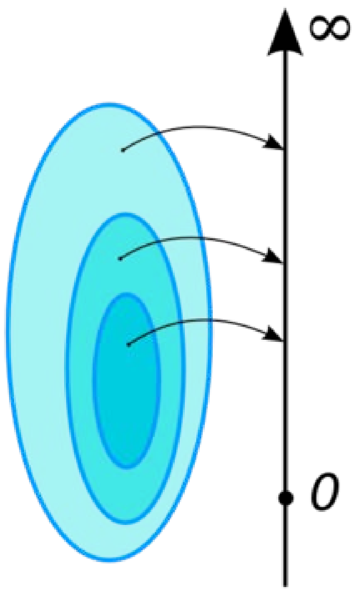
A number of related concepts

- Measure
  - Math
  - Other
- Metric
  - Math
  - Other
- Indicator: same as measure/metric, other
- Measurement

Desperate times call for desperate measures
- English proverb (Hippocrates?)

Measures in mathematics

- Functions from “subsets” to the reals
- A ***measure*** obeys the properties:
    1. Non-negativeness: for any subset X, $f(X) \geq 0$
    2. Null empty set: For the empty set, $f(\emptyset) = 0$
    3. Additivity: for two disjoint subsets X and Y, $f(X \cup Y) = f(X) + f(Y)$



Measures in mathematics

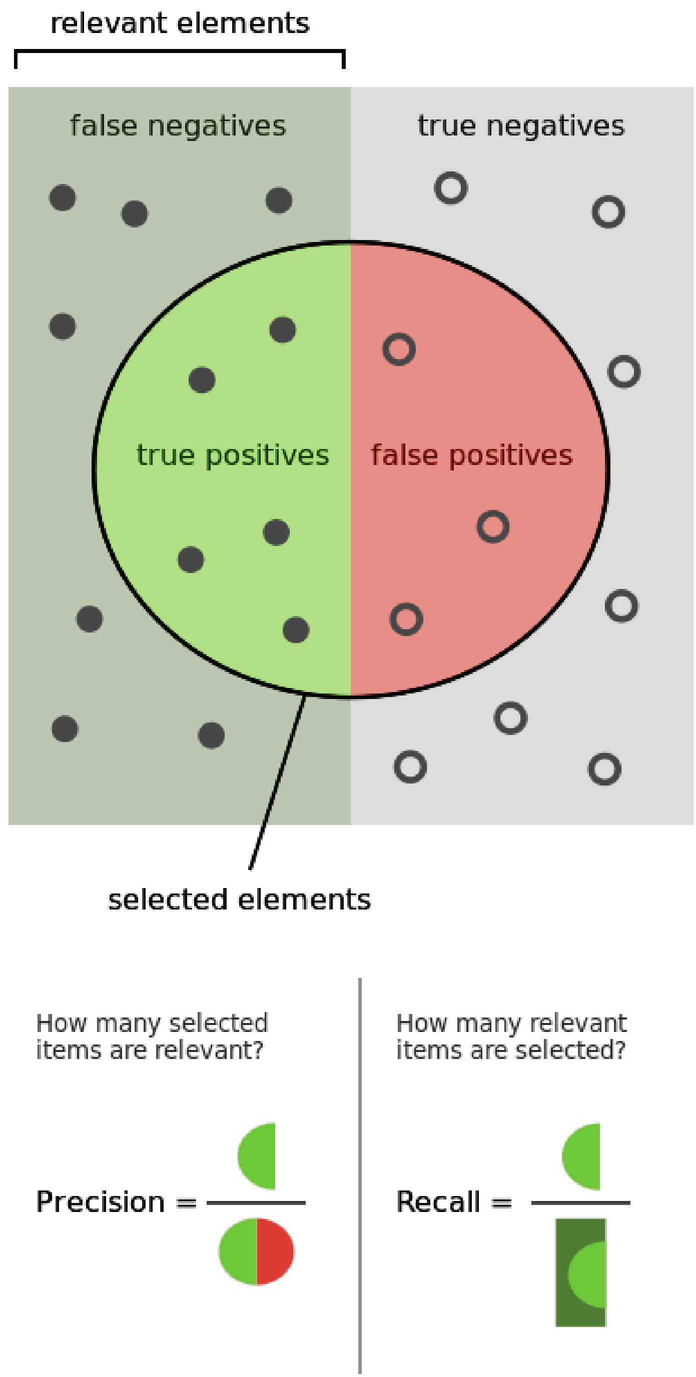
- Example 1: Space is the real line, subsets are disjoint unions of intervals, measure is (total) length
- Example 2: Space is all integers, subsets are finite subsets of integers, measure is number of integers in a subset
- Example 3: Space is outcomes of an experiment (die rolling), measure is probability of the outcome(s)

Measures in the rest of science

- Functions from "something" to the nonnegative reals
- Capture an intuitive aspect: size, quality, difficulty, distance, similarity, usefulness, robustness, … into something well-defined

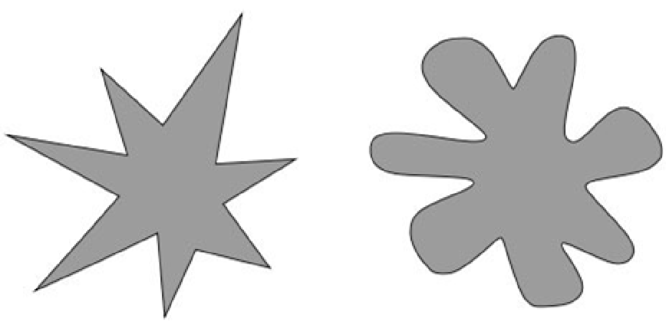
- Precision and recall in information retrieval
- Support and confidence in association rule mining

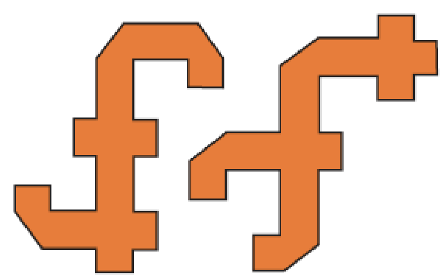
- In the world at large: body mass index, ecological footprint, …

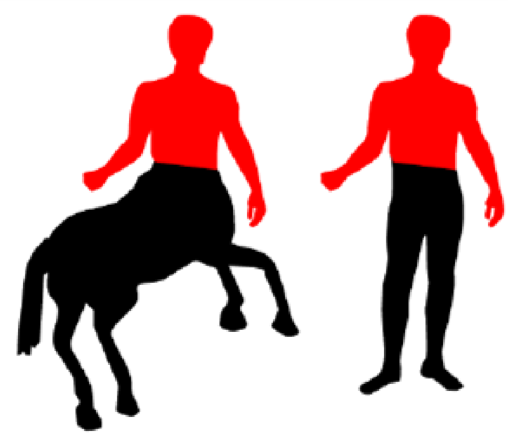

relevant elements
false negatives
true negatives
true positives
false positives
selected elements
How many selected items are relevant?
Precision =
How many relevant items are selected?
Recall =

Measures in the rest of science

- Albeit well-defined, the real connection of the (abstract) function to the intuitive concept is not guaranteed ➔ Needs to be justified or tested
- Example 1: similarity measure for two shapes







Measures in the rest of science

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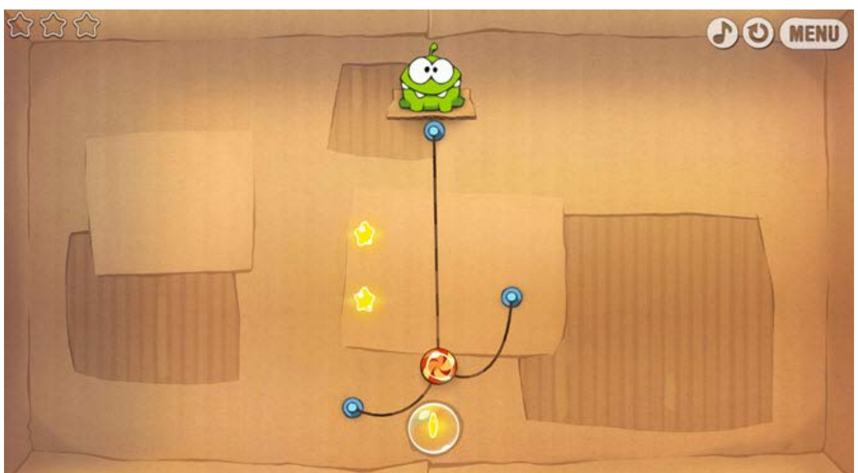
- Example 2: difficulty rating of a level in a puzzle game

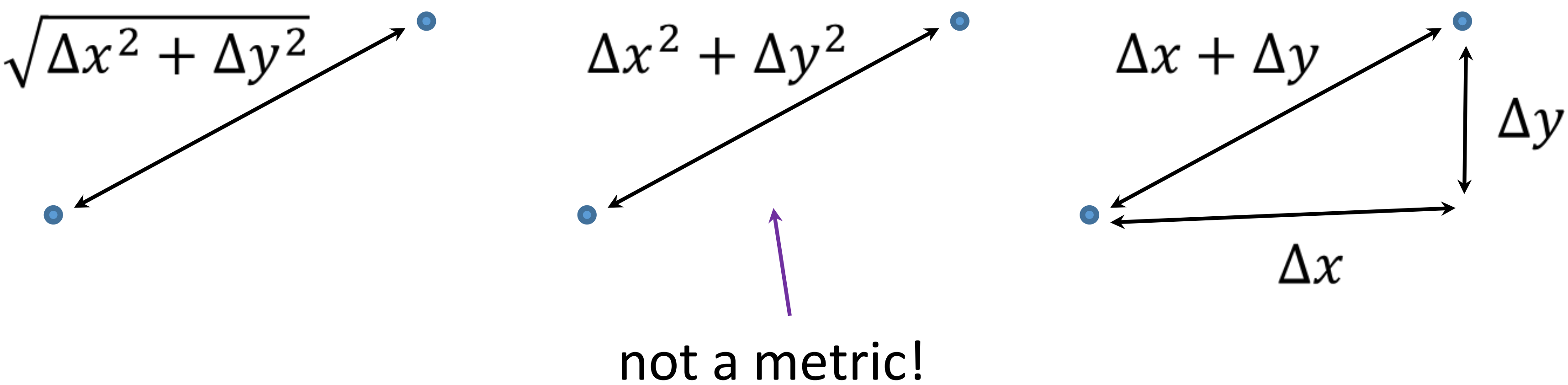


Distance functions, or metrics

- Distance: how far things are apart
- A metric or distance function takes ***two*** arguments and returns a nonnegative real
- Distances on a set X; for any x, y, z in X,
  a ***metric*** is a function $d(x,y) \rightarrow R$ (the reals) where:
  1. $d(x,y) \geq 0$ — non-negative
  2. $d(x,y) = 0$ if and only if $x = y$ — coincidence
  3. $d(x,y) = d(y,x)$ — symmetry
  4. $d(x,z) \leq d(x,y) + d(y,z)$ — triangle inequality

Examples of metrics on points

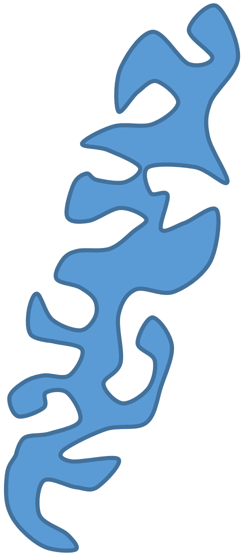
- Euclidean distance on the line, in the plane or in a higher-dimensional space, L_2 distance
  Note: Squared Euclidean distance is not a metric
- City block, Manhattan, or L_1 distance (are the same)
- L_∞ distance (max of differences in the coordinates)



Combining measures

Suppose we have a measure in [0,1] for elongatedness of a shape and another one for frilliness, called E and F

How can we combine these into a score for both elongatedness and frilliness?



Combining measures

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- Weighted linear combination: $\alpha E + (1-\alpha) F$ with $\alpha \in [0,1]$

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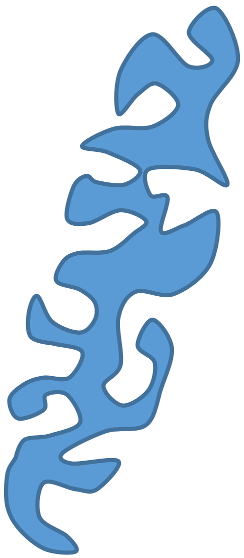
- Weighted linear combination: $\alpha E + (1-\alpha) F$ with $\alpha \in [0,1]$
- Multiplication: $E F$
- Weighted version: $E^{\alpha} F^{1-\alpha}$ with $\alpha \in [0,1]$

Combining measures

Suppose we have a measure in [0,1] for elongatedness of a shape and another one for frilliness, called *E* and *F*

How can we combine these into a score for both elongatedness and frilliness?

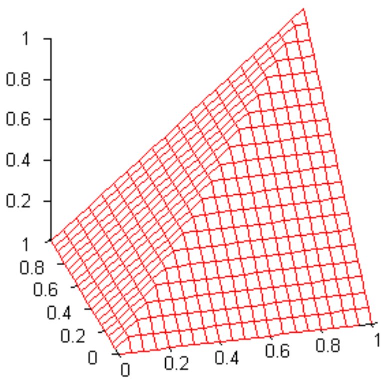
elongated	frilly	combined WLC α = 0.5	combined Mult. α = 0.5
0	0	0	0
1	1	1	1
0	1	0.5	0
0.5	0.5	0.5	0.5
0.5	1	0.75	0.707
0.75	0.75	0.75	0.75

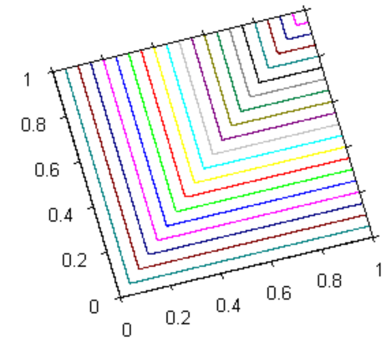


t-norms

- A t-norm is a function T: [0, 1] × [0, 1] → [0, 1] which satisfies the following properties:
  - Commutativity: T(a, b) = T(b, a)
  - Monotonicity: T(a, b) ≤ T(c, d) if a ≤ c and b ≤ d
  - Associativity: T(a, T(b, c)) = T(T(a, b), c)
  - The number 1 acts as identity element: T(a, 1) = a

Minimum t-norm
T(a, b) = min(a, b)





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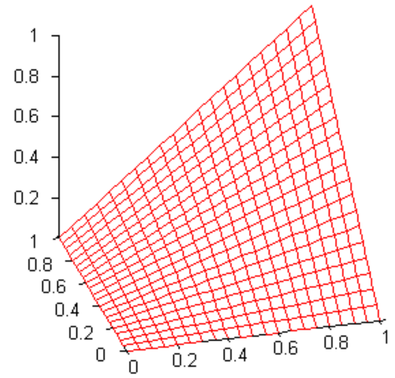
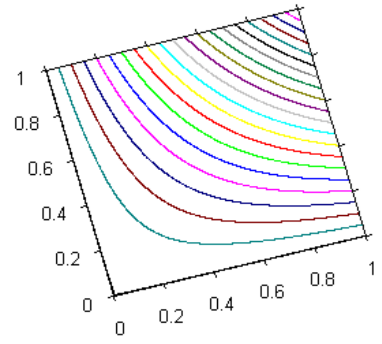
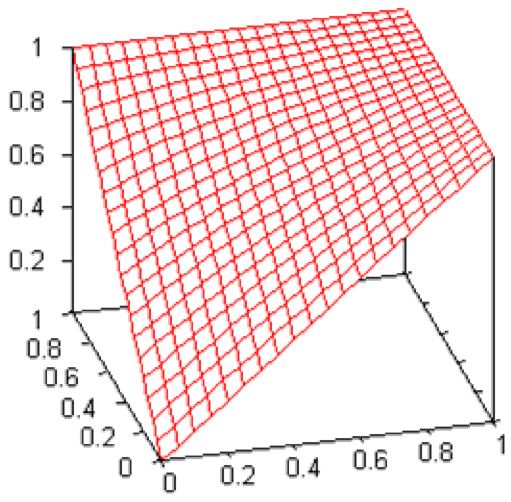
Product t-norm
T(a, b) = ab

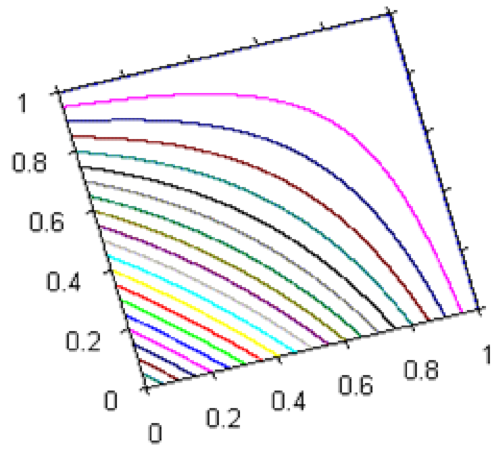




t-conorms

- Similar to t-norms but 0 is the identity: T(a, 0) = a
- Example: Einstein sum T(a, b) = (a + b) / (1 + ab)





Discussion

- Why are measures and metrics useful?
- Do they play a role in fundamental research, in experimental research, and in user study research?
- Measures for what type of concepts?