Fundamental Research

Scientific Perspectives on GMT 2019/2020

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Fundamental Research

- Fully defined problem statement
- In a completely specified framework/setting
- Search for universal truths (no data)
- For GMT: with relevance to games, virtual environments, interaction, visualization, multimedia, …

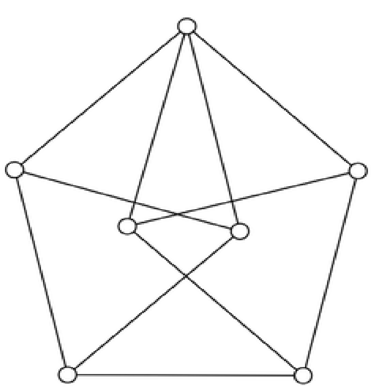
Fundamental research

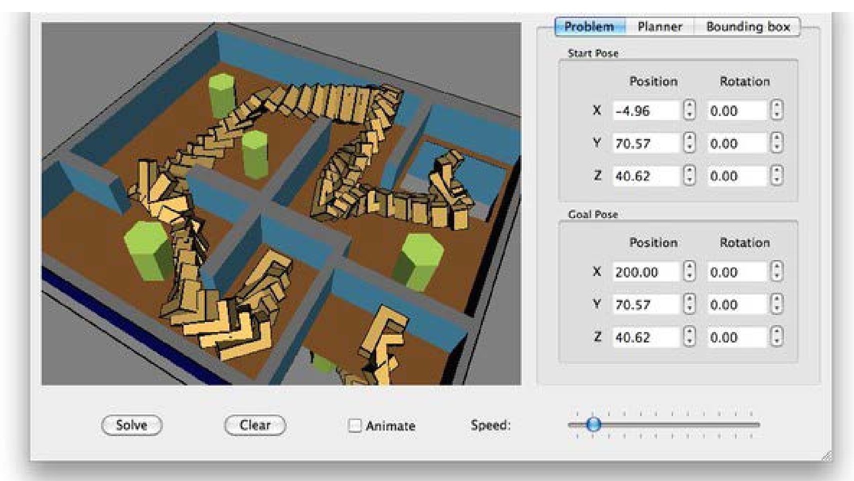
- Research for which data or people are not needed

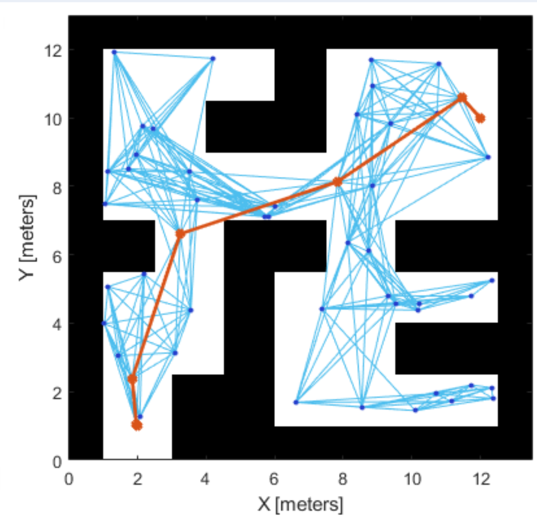
- Synonyms: foundational research, pure research
- Complementary to experimental research, user studies, …

Fundamental research

	Fundamental research question	Applied research question
Fundamental research approach	Can any planar graph with *n* vertices be drawn planarly with all vertices on an *n* x *n* integer grid and straight edges?	Can we compute a motion for a robot amidst polygonal obstacles, if one exists?
Experimental research approach	Can all randomly generated planar graphs with *n* vertices be drawn planarly on an *n* x *n* grid and straight edges?	Does the probabilistic path planner always find a motion for a robot, if one exists?







Types of fundamental research

- Algorithmic efficiency
- Approximation factor
- Competitive ratio
- Probabilistic bounds
- Comparisons on models of computation
- Comparisons of measures or models

Methods of fundamental research

- Proofs
  - By induction
  - By contradiction
  - By construction
  - By algebra (formula manipulation)
- Algorithm design
- Constructions

Formulate results as lemmas or theorems

Interest of fundamental research

- Intrinsic: interesting question on its own, curiosity
- Link to (an abstraction of) an application

Algorithmic efficiency

- Efficiency of algorithms expressed using order notation in size of the input (and sometimes output)
- Complexity classes
  - PSPACE-hardness
  - NP-hardness
  - Polynomial-time solvable
- Efficiency proofs
  - Worst-case upper bounds – all inputs
  - Worst-case lower bounds – one input class

Approximation

- Constant-factor approximation
- PTAS: for any constant $\varepsilon > 0$, there is a polynomial-time $(1+\varepsilon)$-approximation algorithm (running time may depend exponentially on ε, like in $O(2^{\varepsilon} n^2)$)
- LTAS: polynomial → linear (dependency on n)
- FPTAS: running time depends polynomially on n and ε, like in $O(n^2/\varepsilon^3)$
- log n / root(n)-approximation (not constant because approximation quality depends on n, the input size)

Competitive ratio

- For strategies to deal with unknown information
- "the cost of not knowing": how much worse might we do compared to if we knew (in ratio)?

- Most commonly: search strategies where either the target or the environment (or both) is unknown

Searching: competitive analysis

- Best known: finding a door in a wall

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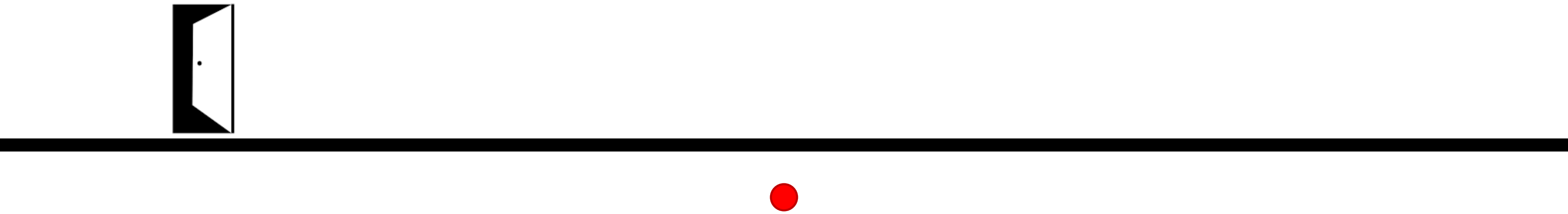

Assume we know the door is *D* away, but we do not know if it is left or right

Searching: competitive analysis

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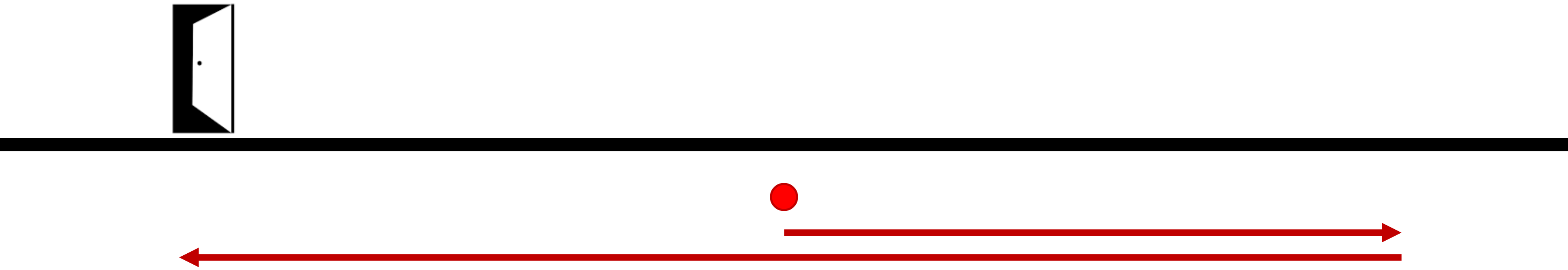

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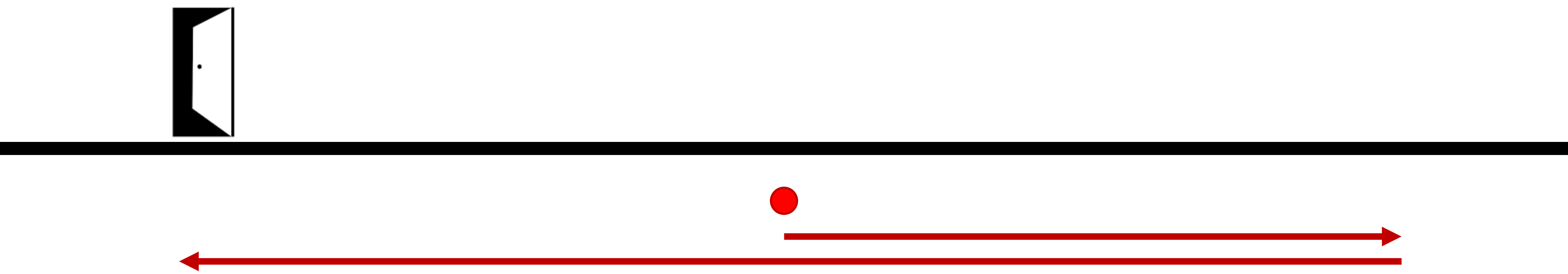


Assume we know the door is *D* away, but we do not know if it is left or right

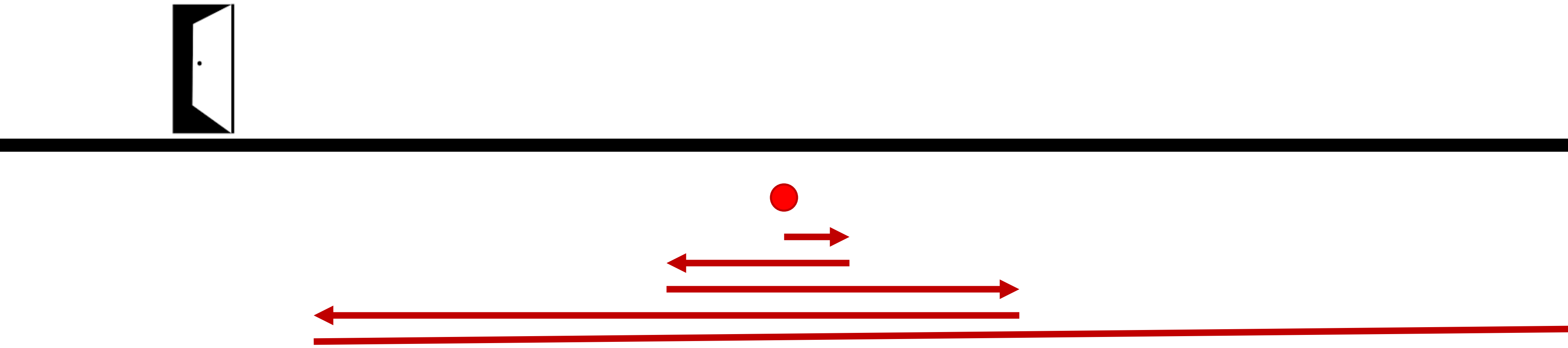
We walk *D* or 3*D*; she who knows, walks only *D*
→ 3-competitive strategy

Searching: competitive analysis

- **Theorem**: Given a searcher looking for a point on the real line who knows the exact distance to the point, there exists a search strategy that is 3-competitive
- The theorem statement assumes certain terms and the model to be known (competitive, precise measurement of walking distance)
- Proof by giving the strategy and analyzing it

Searching: competitive analysis

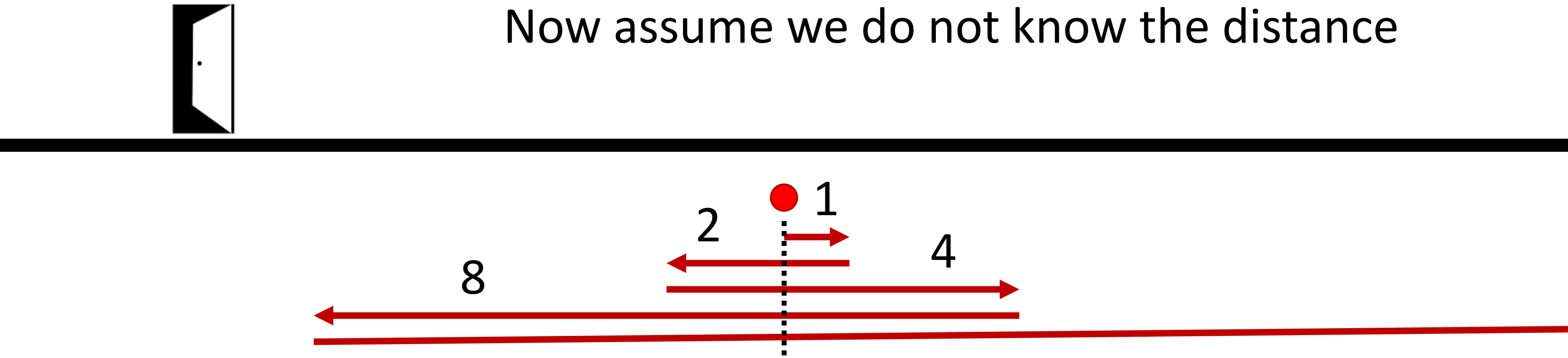
- Best known: finding a door in a wall



Searching: competitive analysis

- Best known: finding a door in a wall

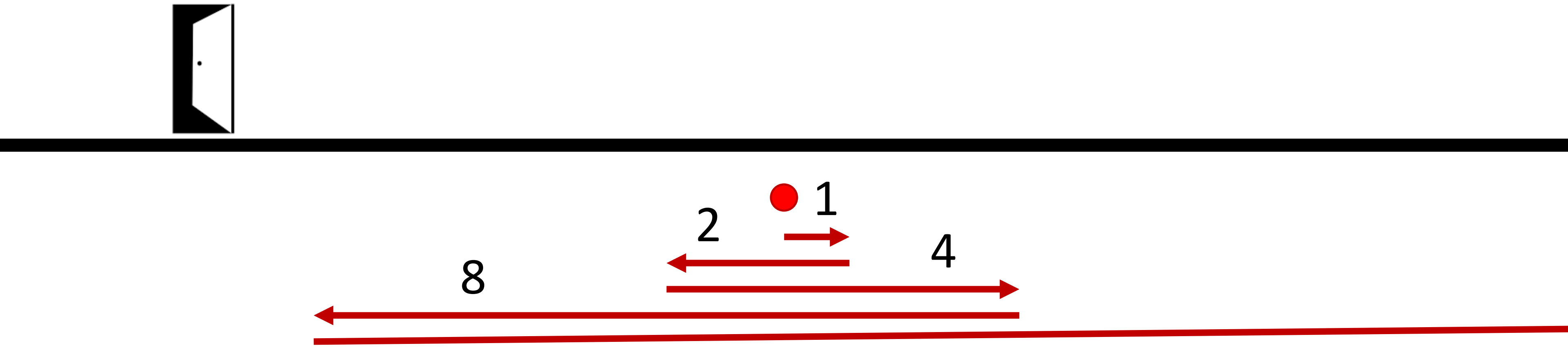
Now assume we do not know the distance



We walk: 1 + (1+2) + (2+4) + (4+8) + …

Searching: competitive analysis

- Best known: finding a door in a wall



Suppose the door is $D = 2^k + d$ away, $0 < d < 2^{k+2}$

We walk: $1 + (1+2) + \ldots + (2^k + 2^{k+1}) + 2^{k+1} + D =$

$2 \times 2^{k+2} - 2 + D < 8 \times 2^k + 2^k + d = 9 \times 2^k + d$

Searching: competitive analysis

- The competitive ratio is

$$\frac{\text{walked by strategy}}{\text{walked by knower}} = \frac{9 \times 2^k + d}{2^k + d} \quad \text{where } 0 < d < 2^{k+2}$$

This is maximized for *d* as small as possible (near zero), in which case the competitive ratio approaches 9; it is always < 9

Searching: competitive analysis

- **Theorem**: Given a searcher looking for a point on the real line, there exists a search strategy that is 9-competitive

- Not quite true ….

Searching: competitive analysis

- **Theorem**: Given a searcher looking for a point on the real line, there exists a search strategy that is 9-competitive, assuming the point is at least some known, arbitrarily small distance away from the starting position

- Proof by giving the strategy and analyzing it

Probabilistic bounds

- Random samples of a data set
- Random sampling of a real-world phenomenon
- Random choices in an algorithm

Models of computation

- What do we allow the machine model (Turing machine?) to do in a unit of time?
  - Any memory look-up or write
  - Any comparison or computation on two reals
  - Any root-finding of a constant-degree polynomial?
  - Trigonometric functions?
  - …
- What do we allow the machine model to do at all?

Models of computation

- What is harder to compute: the area or the perimeter of a simple polygon?

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- Answer 1: Both can be done in O(n) time, if the polygon has n vertices

Models of computation

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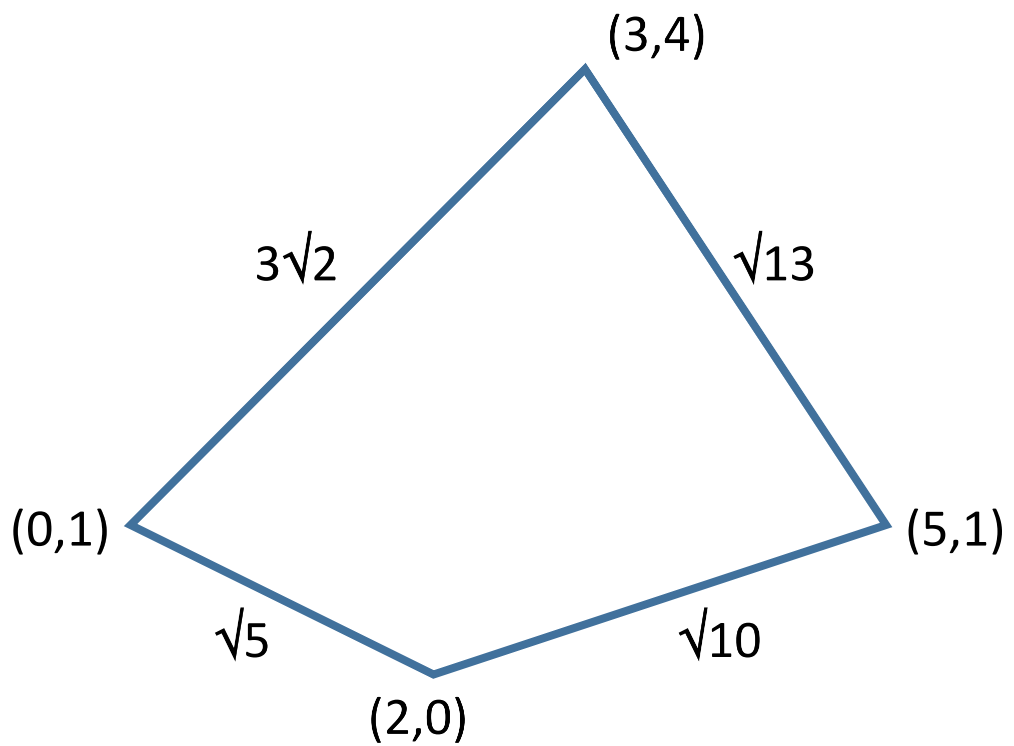
  Assume the vertices are given by integer coordinates

Models of computation

- What is harder to compute: the area or the perimeter of a simple polygon?
  Assume the vertices are given by integer coordinates

- Answer 2:
  - The area can be computed using only +, -, *, and divide by 2 on integers
  - The perimeter requires computing sums of square roots

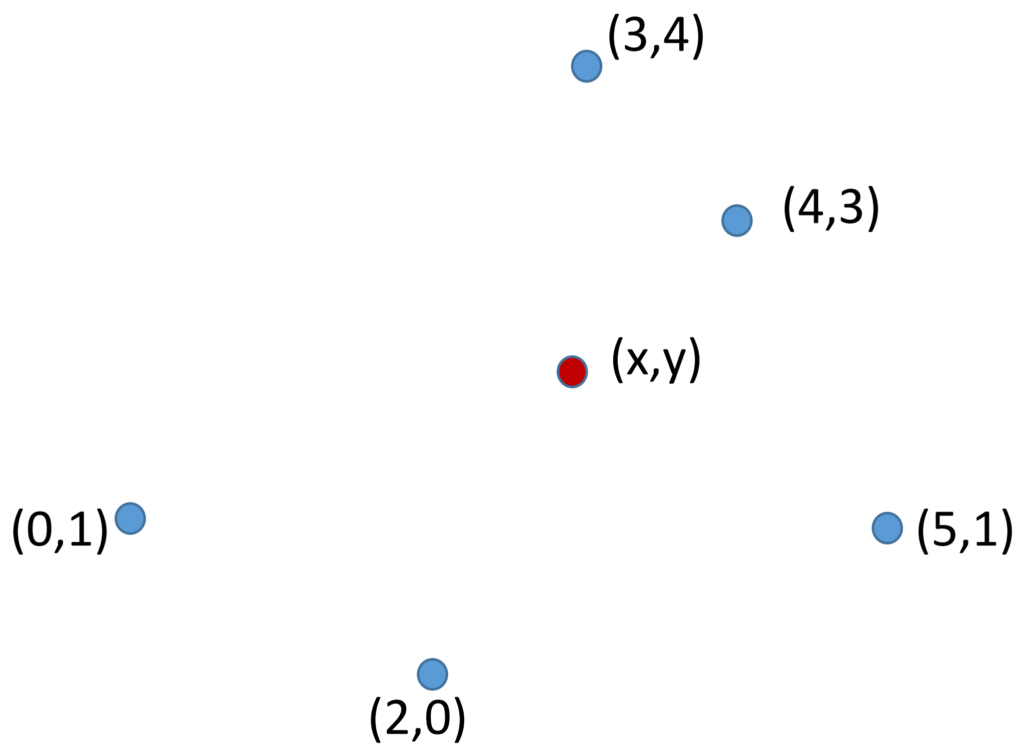


Perimeter = $3\sqrt{2} + \sqrt{13} + \sqrt{10} + \sqrt{5}$

Area = 10

Models of computation

- What is harder to compute: a point that minimizes the **sum of distances** to *n* other points, or a point that minimizes the **sum of squared-distances** to *n* other points?



Minimize:

$$x^2 + (y-1)^2 + (x-2)^2 + y^2 + (x-5)^2 + (y-1)^2 + (x-4)^2 + (y-3)^2 + (x-3)^2 + (y-4)^2$$

$\sqrt{}$ $\sqrt{}$ $\sqrt{}$ $\sqrt{}$ $\sqrt{}$

Models of computation

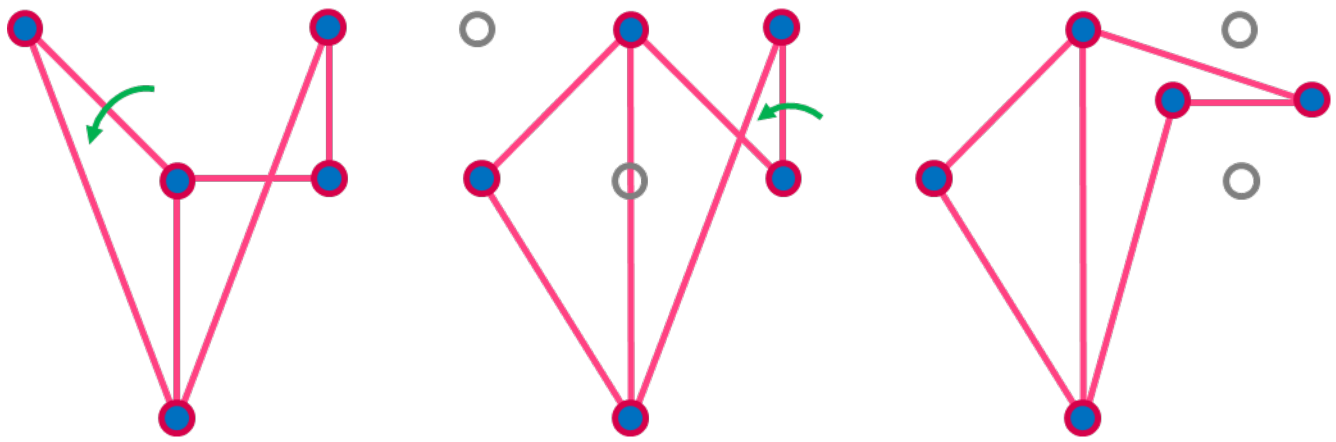
- Minimizing a quadratic function in *x* and *y* is easy, by computing partial derivatives. We need to find the root of a linear function
- Minimizing a function that has the unknowns in several different square roots is hard (even for a single unknown). No closed-form solution exists (a formula like the *abc*-formula or *pq*-formula for finding roots of a quadratic equation)

Models of computation

- What number type is needed to solve a problem?
  - Integers Z
  - Rational numbers (e.g., 2/3) Q
  - Algebraic numbers (e.g., root(2))
  - Transcendental numbers (e.g., π) R
  - Complex numbers (e.g., π + 3i) C

Puzzle game

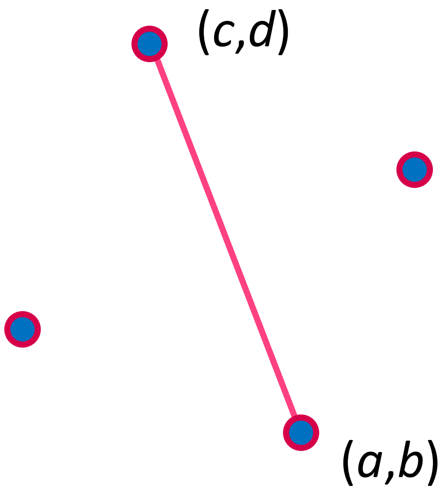
- Example from a puzzle game: 90 degree rotations



Recent master thesis project of Casper van Dommelen

Puzzle game

- In every step we may need extra precision worth one bit, but not more



New coordinates:

((*a*-*b*+*c*+*d*)/2, (*a*+*b*-*c*+*d*)/2)
((*a*+*b*+*c*-*d*)/2, (-*a*+*b*+*c*+*d*)/2)

If *a*, *b*, *c*, and *d* are integer, then the new coordinates may contain halves but not worse → one extra bit needed

(*very easy algebraic proof*)

Puzzle game

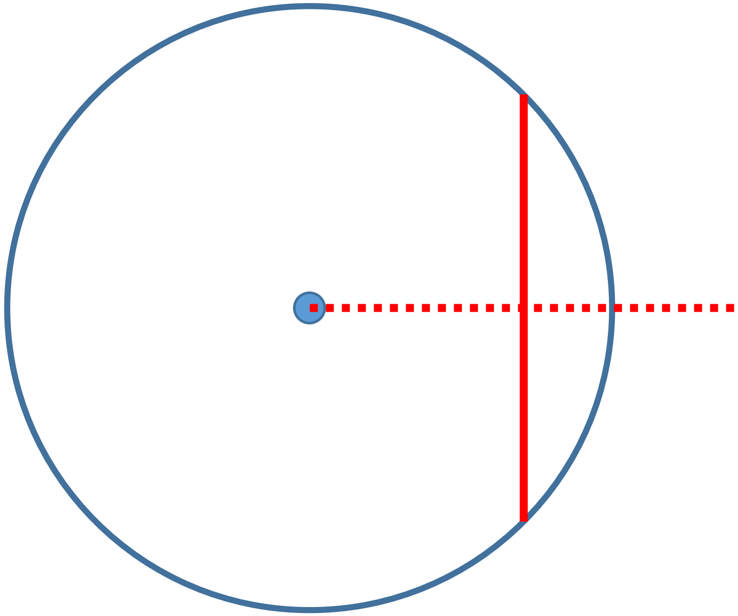
- **Theorem**: for any point set *P*, the center of mass of *P* does not change under a 90 degree rotate of two points of *P* about their midpoint
- Proof by algebra
- **Theorem**: for any point set *P*, the sum of squared distances to the center of mass does not change under a 90 degree rotate of two points of *P* about their midpoint
- Proof by algebra

Puzzle game

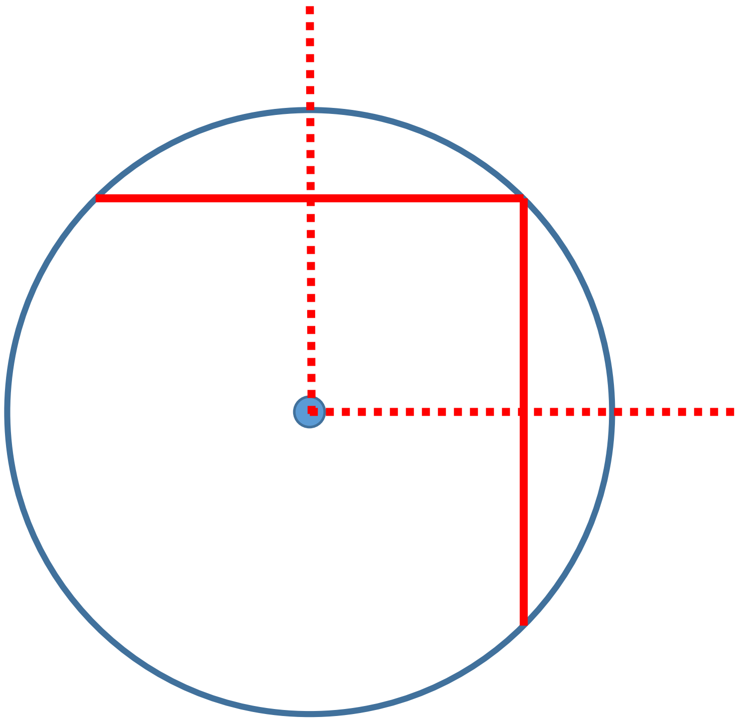
- Fundamental question: Is it possible to define a constant-size play area such that for any graph whose vertices lie in [0,1] x [0,1], any set of rotates stays within that play area?

- Possible answer YES: need proof
- Possible answer NO: need graph (construction) and rotate sequence that gets vertices arbitrarily far away (which is the basis of the proof)

Puzzle game



Puzzle game



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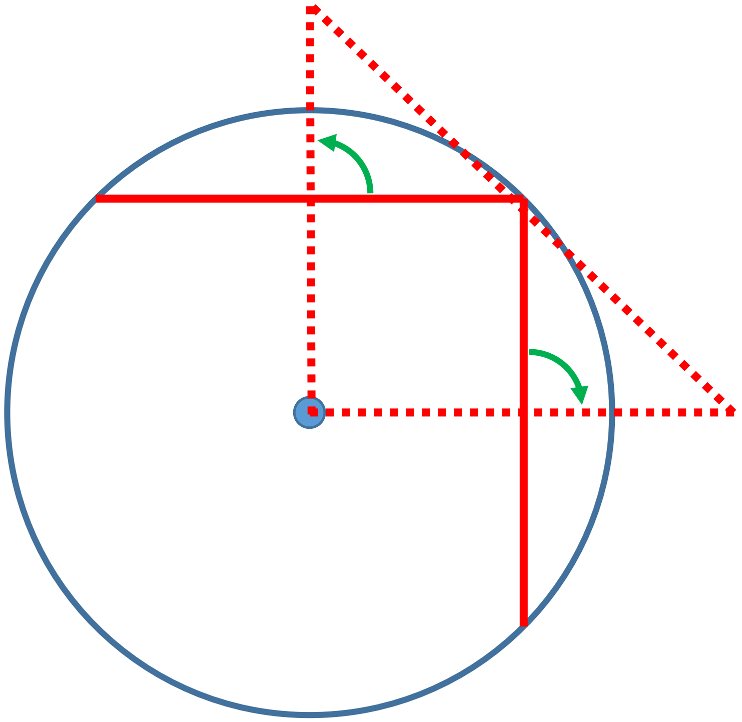
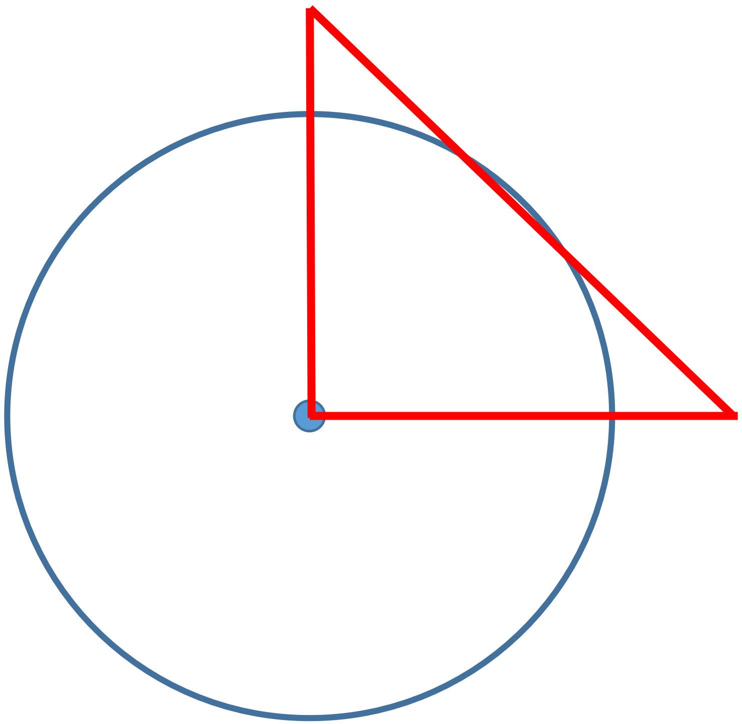


Puzzle game



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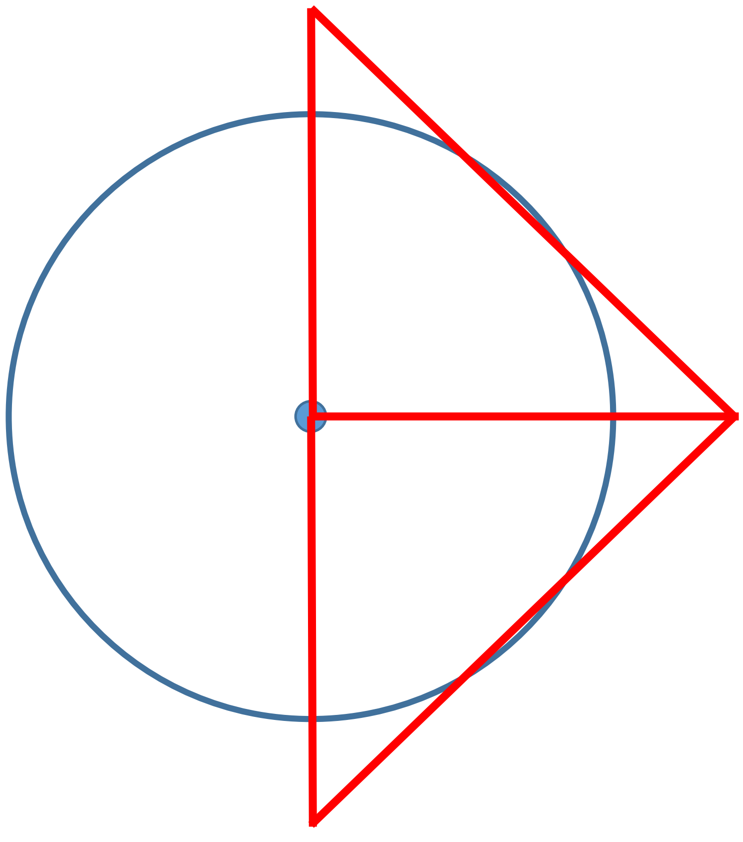

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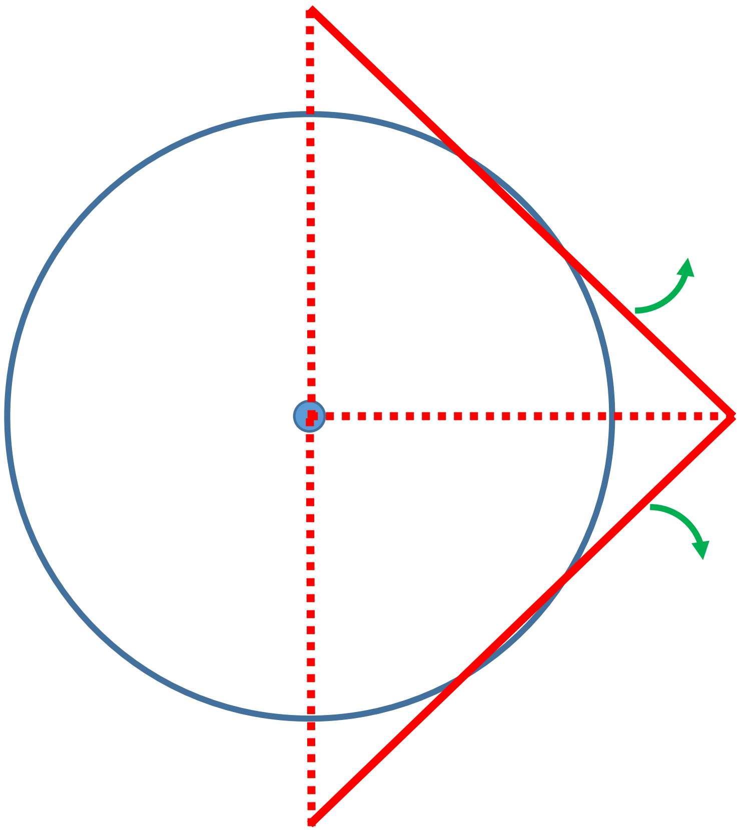
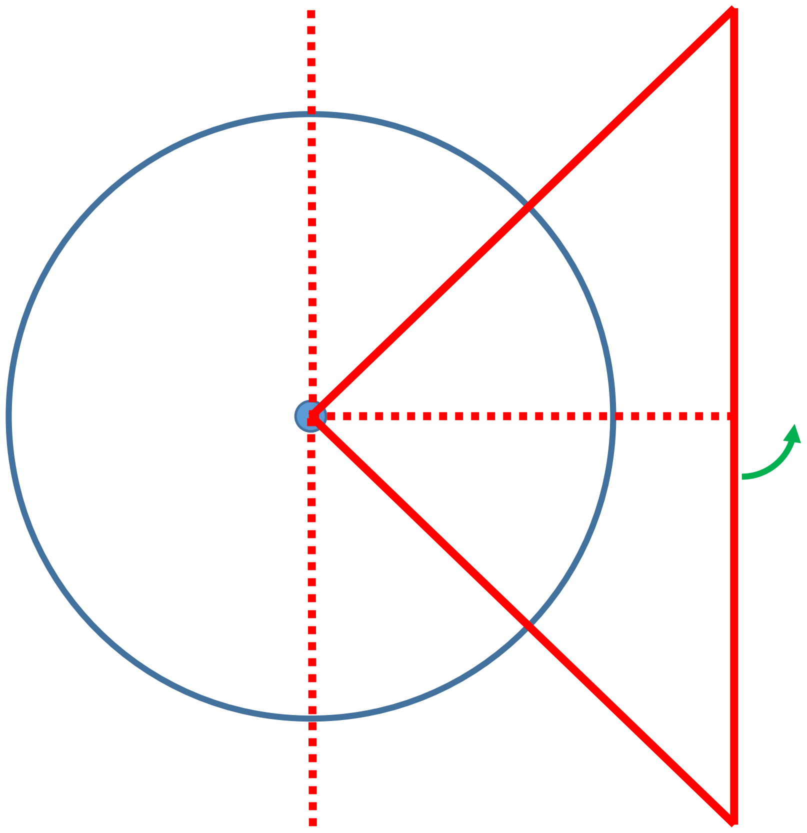


Puzzle game



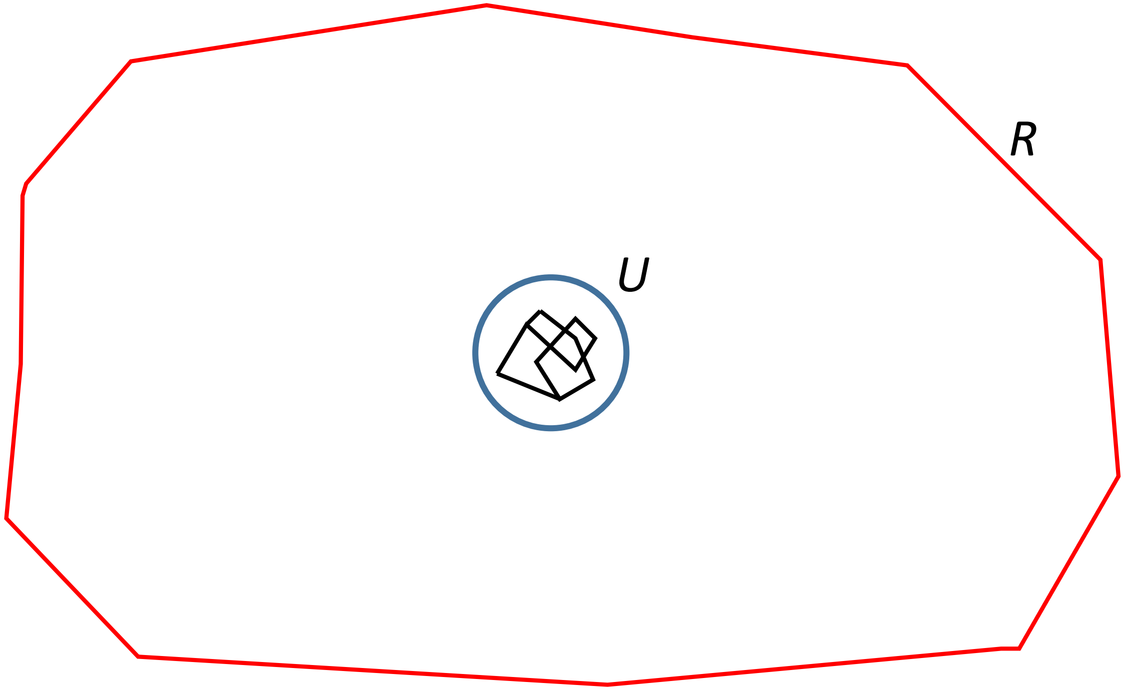
Puzzle game

- Fundamental question: Is it possible to define a constant-size play area such that for any graph whose vertices lie in [0,1] x [0,1], any set of rotates stays within that play area?

- Answer is NO: proof by construction: a graph class and rotate sequence that gets vertices arbitrarily far away

Puzzle game

- **Theorem:** For any bounded region *R* and any unit size region *U* inside it, a graph exists that starts in *U* and can get a vertex outside *R* after finitely many rotations



Puzzle game

- Any single graph cannot give the no-answer (it must be a graph *class*)

- Let *G* be any graph, assume it has *n* vertices, and assume they lie in a radius-1 disk
- Then the sum of squared distances is at most *n*
- So no vertex can be further than root(*n*) away from the center of mass

Puzzle game

- Any single graph cannot give the no-answer (it must be a graph *class*)
  - Let *G* be any graph, assume it has *n* vertices, and assume they lie in a radius-1 disk
  - Then the sum of squared distances is at most *n*
  - So no vertex can be further than root(*n*) away from the center of mass
- For any given graph, *n* is just a value, but a construction that works for *n* arbitrarily large is a graph class

Puzzle game

- The most interesting theoretical question (to me):

  Given any non-planar placement of a connected planar graph, can it always be made planar by rotate-90-degree moves?

- Why is “connected” in the statement?
- Do you see graph subclasses of planar graphs for which you can give an answer?

Puzzle game

- If no, what is the smallest counterexample?
- If yes, can the number of rotates be bounded by a function of *n*, the number of vertices in the graph?

A probabilistic bound

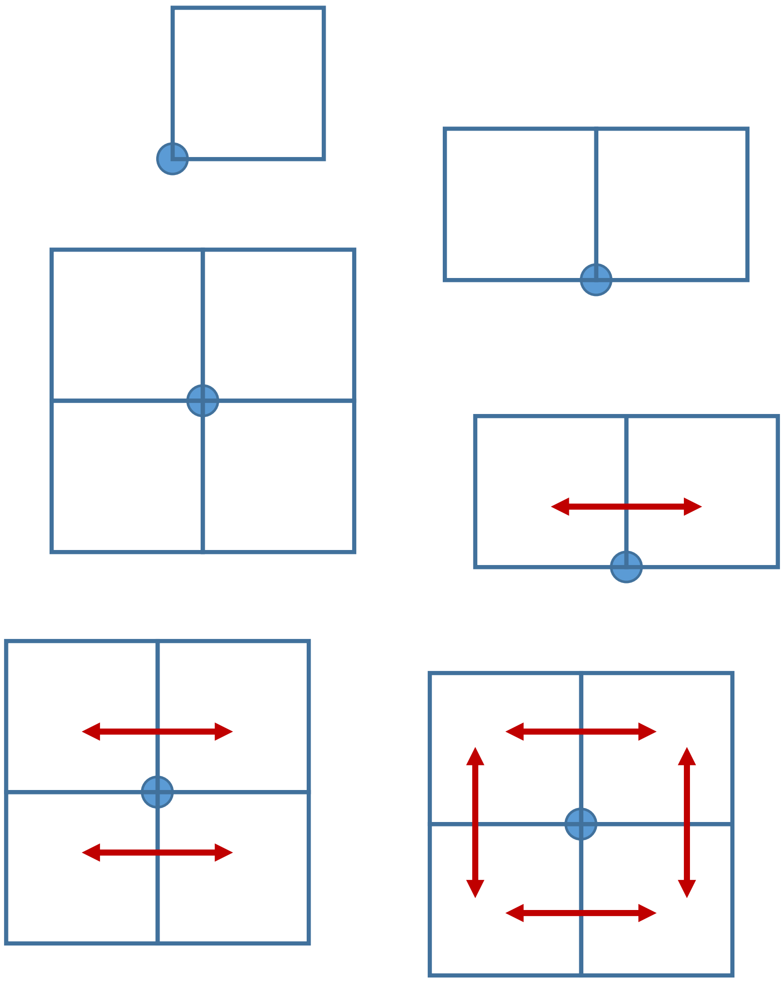
- Suppose *n* numbers are in random order in array A
- We find the max by going from A[1] to A[*n*] and updating *max* when we find a higher number
- What is the expected number of times we update *max*?

A probabilistic bound

- What is the probability that we update *max* after checking A[i]?
- This happens only if A[i] > A[1], …, A[i-1]
- Since A[1], …, A[i] are in random order the probability is 1/i
- The expected number of updates in total is $1/1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots + 1/n < \ln(n) + 1$

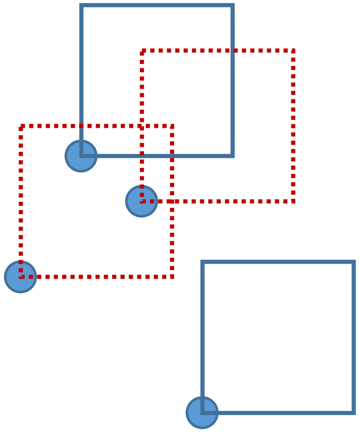
Model comparison

- Text placement on maps:
  - 1-position model
  - 2-position model
  - 4-position model
  - Any c-position model
  - 1-slider model
  - 2-slider model
  - 4-slider model
- All labels are assumed to be unit squares

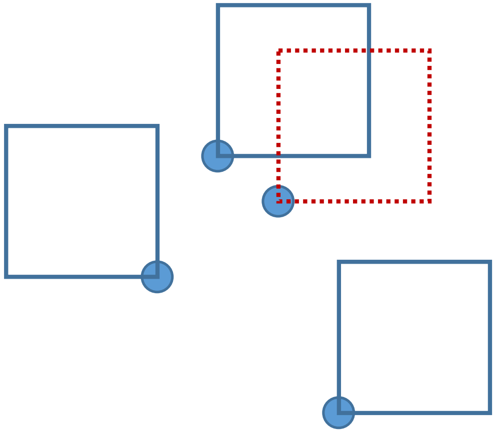


Model comparison

- How much better can one model be than another, in the worst (best?) case for any set of points to be labeled?



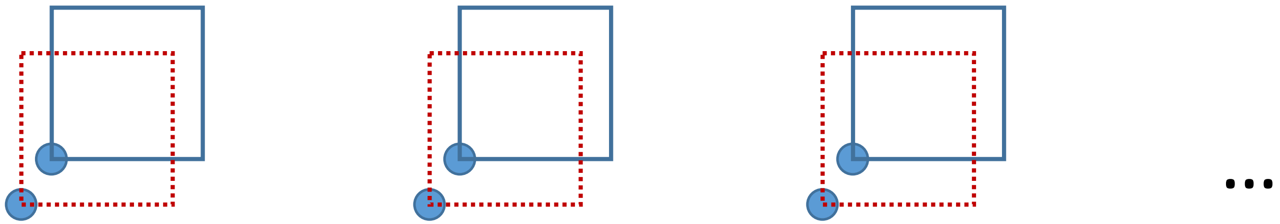
1-position model



2-position model

Model comparison

- The 2-position model can sometimes allow twice as many labels as the 1-position model (for arbitrarily large *n*): by construction of an example class



- Can it be even worse?

Model comparison

- The 2-position model never allows more than twice as many labels as the 1-position model:
  - Take an optimal solution in the 2-position model
  - If at least half the points have their label top-right, then we are done: the 1-position model can choose these
  - Otherwise, at least half the points have their label top-left
  - Choose these points but with a label top-right
  - These are non-intersecting if and only if the top-left ones were, because all labels move exactly one unit to the right

Model comparison

- **Theorem:** For any set of *n* points in the plane, if a disjoint labeling with unit squares exists of a subset of *k* points that are either to the top-left or top-right, then a disjoint labeling with unit squares exists of at least *k*/2 points that are to the top-right

  (For any *n*,) there exists a set of *n* points that allows *n* unit squares to the top-left or top-right, but only n/2 unit squares to the top-right

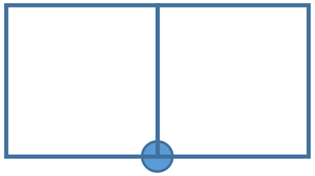
  (The top statement is a *worst-case optimal bound*: We have no hope of placing more than *k*/2 squares in all cases)

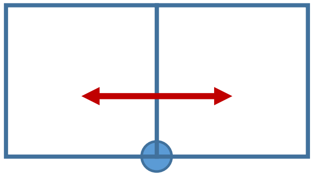
Model comparison

- Similar arguments can be used to compare 1-, 2- and 4-position models

Model comparison

- How do the 2-position model and 1-slider model compare?





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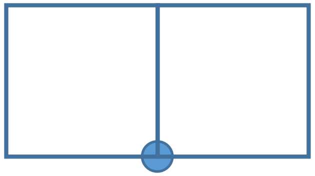
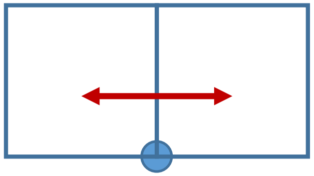
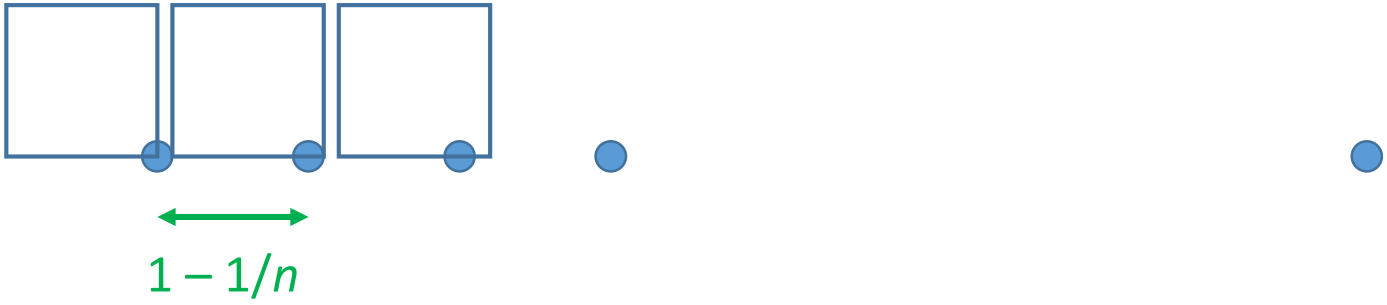




- Sometimes, the 1-slider model allows (nearly) twice as many labels, for any *n*



Model comparison

- This construction (class of points) allows all n points to be labeled in the 1-slider model and only $\lfloor n/2 \rfloor + 1$ points in the 2-position model

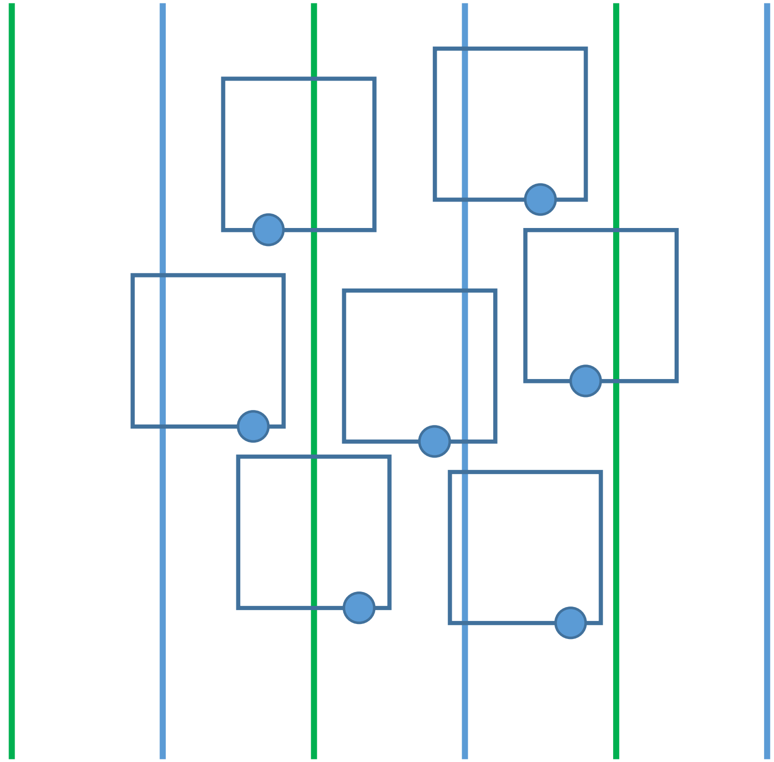
- Can it be worse?

Model comparison

- No, by proof
- Recall labels are unit squares
- Consider the lines x=0, x=1, x=2, x=3, …
- Consider an optimal solution in the 1-slider model: any placed label intersects exactly one line
- Take the labels intersecting the odd or even lines, whichever intersects more labels from the optimal 1-slider solution
  (by pigeonhole principle at least half)

Model comparison

- Slide these labels left or right to a corner position of their square, while still intersecting the same line



Model comparison

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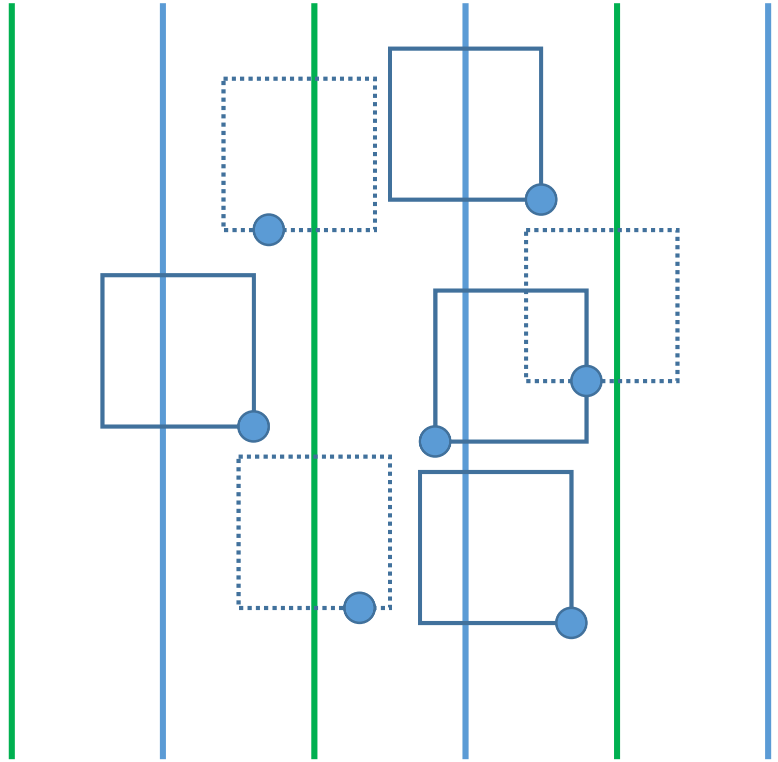

Squares intersecting different blue lines cannot intersect; squares intersecting the same blue line did not intersect in the 1-slider solution

Types of fundamental research

- Algorithmic efficiency – *well known*
- Approximation factor – *well known*
- Competitive ratio – *searching, unknown information*
- Probabilistic bounds – *sampling or randomized algorithms*
- Comparisons on models of computation – *complexity of the basic operations*
- Comparisons of measures or models

Results are given by lemmas and theorems