Syllabus, Chapter 3:

Independences and Graphical Representations

A qualitative notion of independence

Observation:

People are capable of making statements about independences among variables without having to perform numerical calculations.

Conclusion:

In human reasoning behaviour, the qualitative notion of independence is more fundamental than the quantitative notion of independence.

The independence relation of joint distribution $\Pr$

Definition: Let $\boldsymbol{V}$ be a set of random variables and let $\Pr$ be a joint probability distribution on $\boldsymbol{V}$.

The independence relation $I_{\Pr}$ of $\Pr$ is a set $I_{\Pr} \subseteq \mathcal{P}(\boldsymbol{V}) \times \mathcal{P}(\boldsymbol{V}) \times \mathcal{P}(\boldsymbol{V})$, defined for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}$ by

$$(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y}) \in I_{\Pr} \text{ if and only if } \Pr(\boldsymbol{X} \mid \boldsymbol{Y} \wedge \boldsymbol{Z}) = \Pr(\boldsymbol{X} \mid \boldsymbol{Z})$$

Remarks:

- $(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y}) \in I_{\Pr}$ will be written as $I_{\Pr}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$;
  $(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y}) \notin I_{\Pr}$ will be written as $\neg I_{\Pr}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$;
- a statement $I_{\Pr}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ is called an independence statement for the joint distribution $\Pr$.

Properties of I_{Pr}: symmetry

Lemma: $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ if and only if $I_{\mathrm{Pr}}(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{X})$

Proof: [NB This is an exercise from the Syllabus. In proofs you should also explain your steps, like done in the lectures.]

$$
\begin{aligned}
I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y}) &\iff \Pr(\boldsymbol{X} \mid \boldsymbol{Y} \wedge \boldsymbol{Z}) = \Pr(\boldsymbol{X} \mid \boldsymbol{Z}) \\
&\iff \frac{\Pr(\boldsymbol{X} \wedge \boldsymbol{Y} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{Y} \wedge \boldsymbol{Z})} = \frac{\Pr(\boldsymbol{X} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{Z})} \\
&\iff \frac{\Pr(\boldsymbol{X} \wedge \boldsymbol{Y} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{X} \wedge \boldsymbol{Z})} = \frac{\Pr(\boldsymbol{Y} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{Z})} \\
&\iff \Pr(\boldsymbol{Y} \mid \boldsymbol{X} \wedge \boldsymbol{Z}) = \Pr(\boldsymbol{Y} \mid \boldsymbol{Z}) \\
&\iff I_{\mathrm{Pr}}(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{X})
\end{aligned}
$$

■

Properties of $I_{\Pr}$: decomposition

<u>**Lemma**</u>: $I_{\Pr}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y} \cup \boldsymbol{W}) \Rightarrow I_{\Pr}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y}) \ \wedge \ I_{\Pr}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W})$

<u>**Proof**</u>: (sketch) (Note: for $\boldsymbol{U} = \boldsymbol{Y} \cup \boldsymbol{W}$, $c_{\boldsymbol{U}} = c_{\boldsymbol{Y}} \wedge c_{\boldsymbol{W}}$!)
Suppose that

$\Pr(\boldsymbol{X} \mid \boldsymbol{Y} \wedge \boldsymbol{W} \wedge \boldsymbol{Z}) = \Pr(\boldsymbol{X} \mid \boldsymbol{Z})$. Then, by definition,

$$\Pr(\boldsymbol{X} \wedge \boldsymbol{Y} \wedge \boldsymbol{W} \wedge \boldsymbol{Z}) = \Pr(\boldsymbol{Y} \wedge \boldsymbol{W} \wedge \boldsymbol{Z}) \cdot \frac{\Pr(\boldsymbol{X} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{Z})}$$

For $\Pr(\boldsymbol{X} \mid \boldsymbol{Y} \wedge \boldsymbol{Z})$ we find that

$$\begin{aligned} \Pr(\boldsymbol{X} \mid \boldsymbol{Y} \wedge \boldsymbol{Z}) &= \frac{\Pr(\boldsymbol{X} \wedge \boldsymbol{Y} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{Y} \wedge \boldsymbol{Z})} \\ &= \frac{\sum_{c_{\boldsymbol{W}}} \Pr(\boldsymbol{X} \wedge \boldsymbol{Y} \wedge \boldsymbol{Z} \wedge c_{\boldsymbol{W}})}{\Pr(\boldsymbol{Y} \wedge \boldsymbol{Z})} \\ &= \frac{\Pr(\boldsymbol{X} \wedge \boldsymbol{Z})}{\Pr(\boldsymbol{Z})} = \Pr(\boldsymbol{X} \mid \boldsymbol{Z}) \qquad \blacksquare \end{aligned}$$

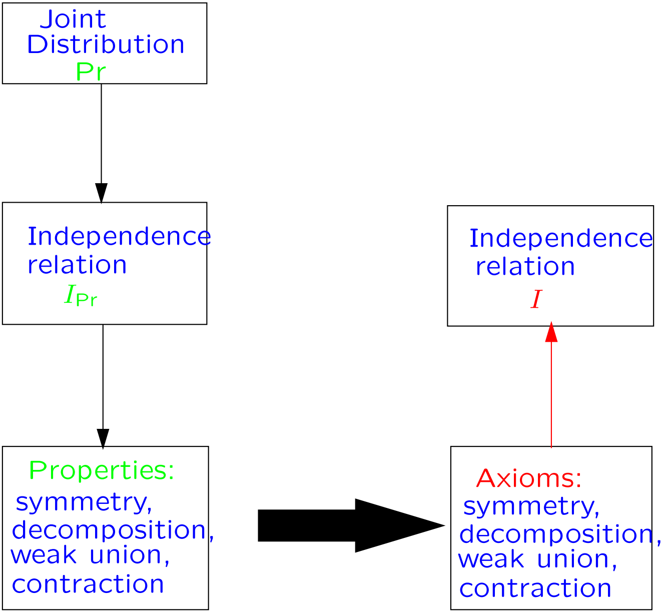
Properties of I_{Pr}: weak union, contraction

Lemma:

- if $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y} \cup \boldsymbol{W})$ then $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z} \cup \boldsymbol{W}, \boldsymbol{Y})$ (weak union);
- if $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W})$ and $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z} \cup \boldsymbol{W}, \boldsymbol{Y})$ then $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y} \cup \boldsymbol{W})$ (contraction)
- (for strictly positive Pr also the intersection property holds; see syllabus)

Proof: left as exercise 3.1.

Defining a qualitative independence relation



The (qualitative) independence relation I

Definition:

Let V be a set of random variables and let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \subseteq \boldsymbol{V}$.

An independence relation I on $\boldsymbol{V}$ is a ternary relation $I \subseteq \mathcal{P}(\boldsymbol{V}) \times \mathcal{P}(\boldsymbol{V}) \times \mathcal{P}(\boldsymbol{V})$ that satisfies the following *axioms*:

1. symmetry:
   if $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ then $I(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{X})$;
2. decomposition:
   if $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y} \cup \boldsymbol{W})$ then $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ and $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W})$;
3. weak union:
   if $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y} \cup \boldsymbol{W})$ then $I(\boldsymbol{X}, \boldsymbol{Z} \cup \boldsymbol{W}, \boldsymbol{Y})$;
4. contraction :
   if $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W})$ and $I(\boldsymbol{X}, \boldsymbol{Z} \cup \boldsymbol{W}, \boldsymbol{Y})$ then $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y} \cup \boldsymbol{W})$.

An example

Lemma:

Let I be an independence relation on a set of random variables $\boldsymbol{V}$. We have that for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \subseteq \boldsymbol{V}$:

if $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ and $I(\boldsymbol{X} \cup \boldsymbol{Z}, \boldsymbol{Y}, \boldsymbol{W})$ then $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W})$

Proof: (Note: we have no $\Pr$, just the axioms!)

We observe that

$$I(\boldsymbol{X} \cup \boldsymbol{Z}, \boldsymbol{Y}, \boldsymbol{W}) \Rightarrow_{\text{symm}} I(\boldsymbol{W}, \boldsymbol{Y}, \boldsymbol{X} \cup \boldsymbol{Z}) \Rightarrow_{\text{weakunion}}$$

$$\Rightarrow I(\boldsymbol{W}, \boldsymbol{Y} \cup \boldsymbol{Z}, \boldsymbol{X}) \Rightarrow_{\text{symm}} I(\boldsymbol{X}, \boldsymbol{Y} \cup \boldsymbol{Z}, \boldsymbol{W})$$

From $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$, $I(\boldsymbol{X}, \boldsymbol{Y} \cup \boldsymbol{Z}, \boldsymbol{W})$ and the contraction axiom we have that $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W} \cup \boldsymbol{Y})$; decomposition now gives $I(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{W})$. ■

Representing independences

Different ways exist of representing an independence relation:

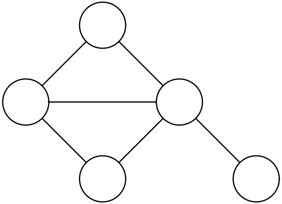
- explicitly list all independence statements of the relation;
- explicitly list only the independence statements of a suitable subset of the relation (the ‘basis’) — all other statements are implicitly represented by means of the axioms;
- code the independence relation in a graph;
- …

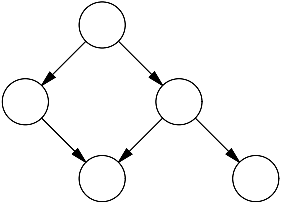
An example

Consider $\boldsymbol{V} = \{V_1, V_2, V_3, V_4\}$ and independence relation I on $\boldsymbol{V}$:

$I(\{V_1\}, \emptyset, \{V_4\})$	$I(\{V_2\}, \emptyset, \{V_1\})$	$I(\{V_4\}, \{V_1\}, \{V_2\})$
$I(\{V_2\}, \emptyset, \{V_4\})$	$I(\{V_1, V_4\}, \emptyset, \{V_2\})$	$I(\{V_4\}, \{V_1\}, \{V_3\})$
$I(\{V_3\}, \emptyset, \{V_4\})$	$I(\{V_2, V_4\}, \emptyset, \{V_1\})$	$I(\{V_4\}, \{V_1\}, \{V_2, V_3\})$
$I(\{V_4\}, \emptyset, \{V_1\})$	$I(\{V_2\}, \emptyset, \{V_1, V_4\})$	$I(\{V_1\}, \{V_2\}, \{V_4\})$
$I(\{V_4\}, \emptyset, \{V_2\})$	$I(\{V_1\}, \emptyset, \{V_2, V_4\})$	$I(\{V_3\}, \{V_2\}, \{V_4\})$
$I(\{V_4\}, \emptyset, \{V_3\})$	$I(\{V_2\}, \{V_1\}, \{V_4\})$	$I(\{V_1, V_3\}, \{V_2\}, \{V_4\})$
$I(\{V_1, V_2\}, \emptyset, \{V_4\})$	$I(\{V_3\}, \{V_1\}, \{V_4\})$	$I(\{V_4\}, \{V_2\}, \{V_1\})$
$I(\{V_1, V_3\}, \emptyset, \{V_4\})$	$I(\{V_2, V_3\}, \{V_1\}, \{V_4\})$	$I(\{V_4\}, \{V_2\}, \{V_3\})$
$I(\{V_2, V_3\}, \emptyset, \{V_4\})$	$I(\{V_4\}, \{V_1, V_2\}, \{V_3\})$	$I(\{V_4\}, \{V_2\}, \{V_1, V_3\})$
$I(\{V_4\}, \emptyset, \{V_1, V_2\})$	$I(\{V_2\}, \{V_1, V_3\}, \{V_4\})$	$I(\{V_1\}, \{V_3\}, \{V_4\})$
$I(\{V_4\}, \emptyset, \{V_1, V_3\})$	$I(\{V_4\}, \{V_1, V_3\}, \{V_2\})$	$I(\{V_2\}, \{V_3\}, \{V_4\})$
$I(\{V_4\}, \emptyset, \{V_2, V_3\})$	$I(\{V_1\}, \{V_2, V_3\}, \{V_4\})$	$I(\{V_1, V_2\}, \{V_3\}, \{V_4\})$
$I(\{V_1, V_2, V_3\}, \emptyset, \{V_4\})$	$I(\{V_4\}, \{V_2, V_3\}, \{V_1\})$	$I(\{V_1\}, \{V_4\}, \{V_2\})$
$I(\{V_4\}, \emptyset, \{V_1, V_2, V_3\})$	$I(\{V_4\}, \{V_3\}, \{V_1, V_2\})$	$I(\{V_2\}, \{V_4\}, \{V_1\})$
$I(\{V_1\}, \emptyset, \{V_2\})$	$I(\{V_4\}, \{V_3\}, \{V_1\})$	$I(\{V_3\}, \{V_1, V_2\}, \{V_4\})$

Coding an independence relation with a graph





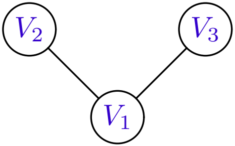
The global idea of coding an independence relation I on $\boldsymbol{V}$ in graph G:

- each variable $V_i \in \boldsymbol{V}$ corresponds to a node $V_i \in \boldsymbol{V}_G$;
- (combinations of) edges/arcs define a graphical notion of (d-)separation;
- there exists a mapping between (d-)separation and relation I

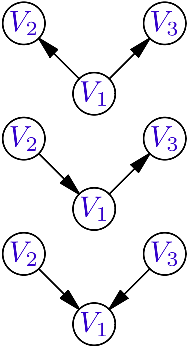
In the lectures, we from here on focus on $I_{\mathrm{Pr}} \subset I$.

Graphs as a language for coding I

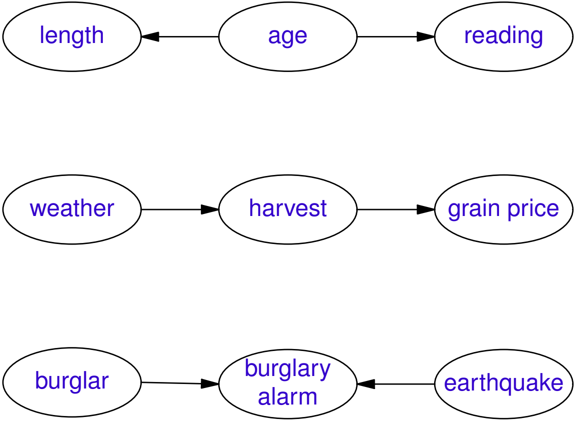
The directed graph provides a more expressive language than the undirected graph:



vs.



Example reasoning patterns



- do arcs capture causality? not necessarily
- do reasoning patterns really differ? last differs from first two
- what if we drop the direction of the arcs? last one incomplete

“Intercausal” interaction

Consider an experiment with two coins and a bell: the bell sounds iff the two coins have the same outcome after a toss.

Consider: variable C_1: the outcome of tossing coin one;
variable C_2: the outcome of tossing coin two;
variable B: whether or not the bell sounds;

independence relation I for this experiment.

We have, among others, that

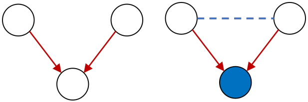
$$I_{\mathrm{Pr}}(\{C_1\}, \emptyset, \{C_2\}) \quad \neg I_{\mathrm{Pr}}(\{C_1\}, \{B\}, \{C_2\})$$
$$I_{\mathrm{Pr}}(\{C_1\}, \emptyset, \{B\}) \quad \neg I_{\mathrm{Pr}}(\{C_1\}, \{C_2\}, \{B\})$$
$$I_{\mathrm{Pr}}(\{C_2\}, \emptyset, \{B\}) \quad \neg I_{\mathrm{Pr}}(\{C_2\}, \{C_1\}, \{B\})$$

This independence relation is an example of an independence relation with what is called an induced ‘dependency’.

Directed versus undirected graph language

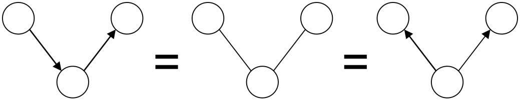
Arc directions encode the possibility of induced dependencies:

- a head-to-head node captures the possible occurrence of an "intercausal" interaction between its parents:



Note:

An acyclic(!) directed graph without head-to-head nodes encodes the same independences as its underlying undirected graph:



Undirected graphs:
separation and (in)dependence

The lectures and the examination focus on directed graphical models. However, the concept of separation in undirected graphs is easier than the concept of d-separation in directed graphs, so you may want to study undirected graphs first. Both course syllabus and these slides (with accompanying video) provide the necessary information and exercises for that.

Skip to directed graphs

The separation criterion: introduction

Definition:
Let $G = (\boldsymbol{V}_G, \boldsymbol{E}_G)$ be an undirected graph with edges $\boldsymbol{E}_G$ and nodes $\boldsymbol{V}_G = \{V_1, \ldots, V_n\}$, $n > 1$.

Let s be a path in G from a node V_i to a node V_j.

The path s is blocked by a set of nodes $\boldsymbol{Z} \subseteq \boldsymbol{V}_G$, if at least one node from $\boldsymbol{Z}$ is *on* the path s.

If s is not blocked by $\boldsymbol{Z}$, the path is called active given $\boldsymbol{Z}$.

The separation criterion

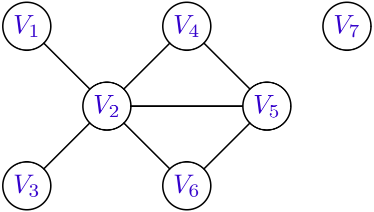
Definition:
Let $G = (\boldsymbol{V}_G, \boldsymbol{E}_G)$ be an undirected graph. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}_G$ be sets of nodes in G.

The set $\boldsymbol{Z}$ separates the set $\boldsymbol{X}$ from $\boldsymbol{Y}$ in G— Notation: $\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G$— if *every simple* path in G from a node in $\boldsymbol{X}$ to a node in $\boldsymbol{Y}$ is *blocked* by $\boldsymbol{Z}$.

Remarks:

- the above notion is known as the separation criterion for undirected graphs;
- if there is no path between the nodes $\boldsymbol{X}$ and $\boldsymbol{Y}$ in a graph G, then $\langle \boldsymbol{X} \mid \emptyset \mid \boldsymbol{Y} \rangle_G$.

An example



Which of the following separation statements are valid?

a) $\langle \{V_1\} \mid \{V_2\} \mid \{V_3, V_6\} \rangle_G$

b) $\langle \{V_4\} \mid \{V_2, V_5\} \mid \{V_6\} \rangle_G$

c) $\langle \{V_4\} \mid \{V_1, V_2, V_5\} \mid \{V_6\} \rangle_G$

d) $\langle \{V_1\} \mid \{V_4\} \mid \{V_5\} \rangle_G$

e) $\langle \{V_1, V_5, V_6\} \mid \emptyset \mid \{V_7\} \rangle_G$

f) $\langle \{V_2\} \mid \{V_5\} \mid \{V_7\} \rangle_G$

g) $\langle \{V_1\} \mid \{V_5\} \mid \{V_2\} \rangle_G$

Answer: all except d) and g)

Independence relations and undirected graphs

Definition: Let I be an independence relation on a set of random variables $\boldsymbol{V}$. Let $G = (\boldsymbol{V}_G, \boldsymbol{E}_G)$ be an undirected graph with $\boldsymbol{V}_G = \boldsymbol{V}$.

- graph G is called a dependency map (D-map) for I_{Pr} if for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}$ we have:

  if $I_{\text{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ then $\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G$;

- graph G is called an independency map (I-map) for I_{Pr} if for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}$ we have:

  if $\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G$ then $I_{\text{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$;

- graph G is called a perfect map (P-map) for I_{Pr} if G is both a dependency map and an independency map for I_{Pr}.

undirected D-maps: what can they tell?

Let I_{Pr} be an independence relation and G an undirected graph.

Consider a D-map for I_{Pr}, then

V_1 and V_2 neighbours[2] $\Longrightarrow$ V_1, V_2 dependent

V_1 and V_2 non-neighbours $\Longrightarrow$?? V_1 and V_2 can be:

dependent,

independent, *or*

conditionally independent

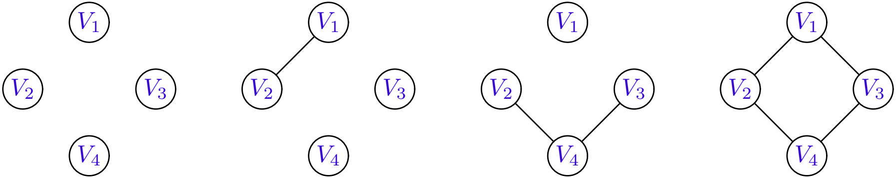
[2] i.e. directly connected by an edge

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$, defined by

$$I_{\mathrm{Pr}}(\{V_1\}, \{V_2, V_3\}, \{V_4\}) \text{ and } I_{\mathrm{Pr}}(\{V_2\}, \{V_1, V_4\}, \{V_3\})$$

Which of the following undirected graphs are examples of D-maps for I_{Pr} ?



See Exercise 3.5 (Syllabus) Answer: all

Undirected I-maps: what can they tell?

Let I_{Pr} be an independence relation and G an undirected graph.

Consider an I-map for I_{Pr}, then

V_1 and V_2 non-neighbours $\Longrightarrow$ V_1, V_2 (condit.) independent

V_1 and V_2 neighbours $\Longrightarrow$?? V_1 and V_2 can be

dependent,

independent, *or*

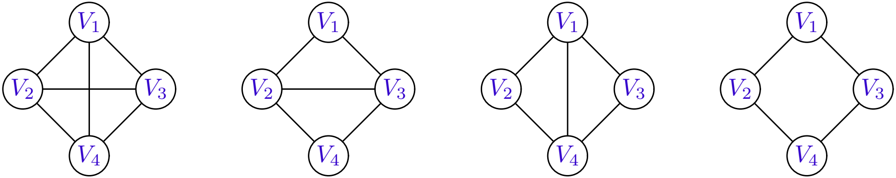
conditionally independent

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$, defined by

$$I_{\mathrm{Pr}}(\{V_1\}, \{V_2, V_3\}, \{V_4\}) \text{ and } I_{\mathrm{Pr}}(\{V_2\}, \{V_1, V_4\}, \{V_3\})$$

Which of the following undirected graphs are examples of I-maps for I_{Pr} ?



(See Exercise 3.5) Answer: all

Properties of I

Let I be an independence relation on a set of random variables $\boldsymbol{V}$.

Lemma:
Every independence relation I has an undirected D-map.

Proof:
The undirected graph $G = (\boldsymbol{V}, \emptyset)$ is a D-map for I. ■

Lemma:
Every independence relation I has an undirected I-map.

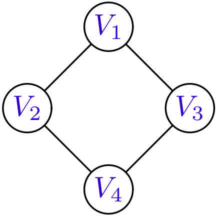
Proof:
The undirected graph $G' = (\boldsymbol{V}, \boldsymbol{V} \times \boldsymbol{V})$ is an I-map for I. ■

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$, defined by

$$I_{\mathrm{Pr}}(\{V_1\}, \{V_2, V_3\}, \{V_4\}) \text{ and } I_{\mathrm{Pr}}(\{V_2\}, \{V_1, V_4\}, \{V_3\})$$

The following undirected graph is a perfect map for I_{Pr}:



Is this P-map unique ?

Does every I_{Pr} have a P-map ?

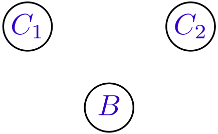
Answer: yes

Answer: no, see next slide

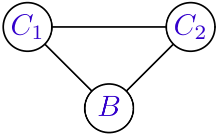
An example

Reconstider the experiment with the two coins and the bell.

- the following graph is the only D-map for the independence relation I_{Pr} of this experiment:



- the following graph is the only I-map for I_{Pr}:



- I_{Pr} does not have a perfect map !

Directed acyclic graphs:

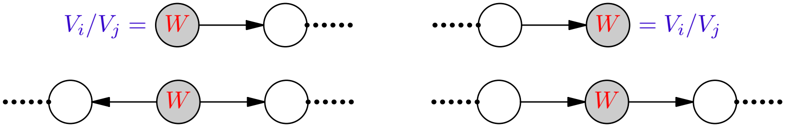
d-separation and (in)dependence

The d-separation criterion: introduction

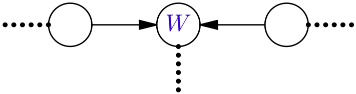
Definition: Let $G = (\boldsymbol{V}_G, \boldsymbol{A}_G)$ be an acyclic directed graph (DAG), and let s be a chain in G between V_i and $V_j \in \boldsymbol{V}_G$.

Chain s is blocked (or: in-active) by a set $\boldsymbol{Z} \subseteq \boldsymbol{V}_G$ if s contains a node W for which one of the following holds:

- $W \in \boldsymbol{Z}$ and W has at most one incoming arc *on* chain s:

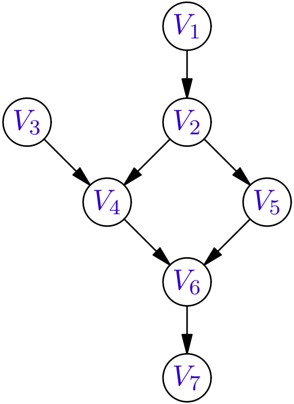


- $\boldsymbol{\sigma}^*(W) \cap \boldsymbol{Z} = \emptyset$ and W has two incoming arcs *on* chain s:



An example

Consider the following DAG and some of its chains:



1) V_4, V_2, V_5 from V_4 to V_5
2) V_1, V_2, V_5, V_6, V_7 from V_1 to V_7
3) V_3, V_4, V_6, V_5 from V_3 to V_5
4) V_2, V_4 from V_2 to V_4

Which chains are blocked by which of the following sets?

$\emptyset, \{V_2\}, \{V_5\}, \{V_2, V_5\},$
$\{V_4\}, \{V_6\}, \{V_4, V_6\}, \{V_7\}$

Answers: $\emptyset$:3; $\{V_2\}$: all; $\{V_5\}$: 1,2,3; $\{V_2, V_5\}$: all; $\{V_4\}$: 1,3,4; $\{V_6\}$: 2; $\{V_4, V_6\}$: all; $\{V_7\}$: 2

The d-separation criterion

Definition:
Let $G = (V_G, A_G)$ be an acyclic directed graph. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq V_G$ be sets of nodes in G.

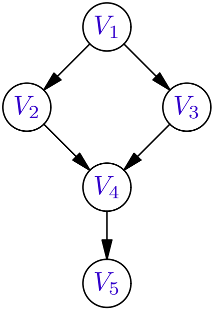
The set $\boldsymbol{Z}$ d-separates $\boldsymbol{X}$ from $\boldsymbol{Y}$ in G—notation: $\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G^d$ —if *every simple chain* in G from a node in $\boldsymbol{X}$ to a node in $\boldsymbol{Y}$ is blocked by $\boldsymbol{Z}$.

Remarks:

- The above notion is known as the d-separation criterion;
- $\langle \boldsymbol{X} \mid \emptyset \mid \boldsymbol{Y} \rangle_G^d$ indicates that all chains between $\boldsymbol{X}$ and $\boldsymbol{Y}$, if any, contain a head-to-head node;
- if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are not d-separated by $\boldsymbol{Z}$, we say that they are d-connected given $\boldsymbol{Z}$; the chain(s) between $\boldsymbol{X}$ and $\boldsymbol{Y}$ that are not blocked are said to be active given $\boldsymbol{Z}$.

An example

Consider the following DAG and d-separation statements:



a) $\langle \{V_1\} \mid \{V_2, V_3\} \mid \{V_5\} \rangle_G^d$

b) $\langle \{V_1\} \mid \{V_4\} \mid \{V_5\} \rangle_G^d$

c) $\langle \{V_2\} \mid \{V_1\} \mid \{V_3\} \rangle_G^d$

d) $\langle \{V_2\} \mid \{V_1, V_5\} \mid \{V_3\} \rangle_G^d$

e) $\langle \{V_2\} \mid \emptyset \mid \{V_3\} \rangle_G^d$

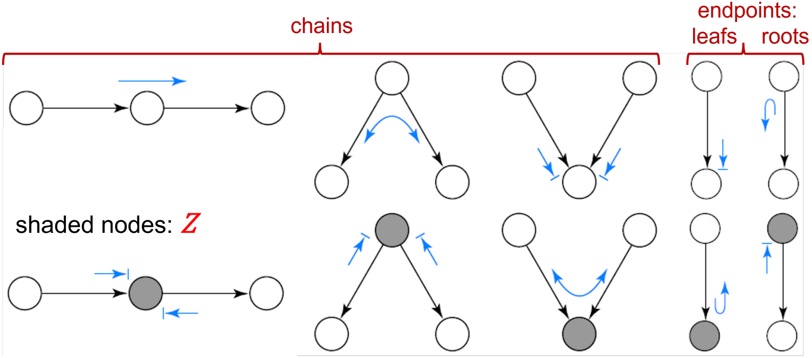
f) $\langle \{V_1\} \mid \{V_3, V_4\} \mid \{V_2\} \rangle_G^d$

Which d-separation statements are valid in the graph ?

Answer: a,b,c

Bayes-Ball for determining d-separation

$\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G^d$? Drop bouncing balls at $\boldsymbol{X}$, which bounce from node to node along chains, using the 10 rules of Bayes-ball:



- balls quit a chain at a stop →|
- any node reached (visited) by a ball is on an active chain
- $\boldsymbol{Y}$ consists of all non-visited nodes

Independence relations and directed graphs

Definition:
Let I_{Pr} be an independence relation on a set of random variables $\boldsymbol{V}$.
Let $G = (\boldsymbol{V}_G, \boldsymbol{A}_G)$ be an acyclic directed graph with $\boldsymbol{V}_G = \boldsymbol{V}$.

- graph G is called a dependency map (D-map) for I_{Pr} if for every $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}$ we have that:

  if $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$ then $\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G^d$;

- graph G is called an independency map (I-map) for I_{Pr} if for every $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}$ we have that:

  if $\langle \boldsymbol{X} \mid \boldsymbol{Z} \mid \boldsymbol{Y} \rangle_G^d$ then $I_{\mathrm{Pr}}(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{Y})$;

- graph G is called a perfect map (P-map) for I_{Pr} if G is both a D-map and an I-map for I_{Pr}.

Directed D-maps: what can they tell?

Let I_{Pr} be an independence relation and G a DAG.

Consider a D-map for I_{Pr}, then

V_1 and V_2 neighbours[3] $\Longrightarrow$ V_1, V_2 dependent

V_1 and V_2 non-neighbours $\Longrightarrow$?? V_1 and V_2 can be:

dependent,

independent,

conditionally dependent, *or*

conditionally independent

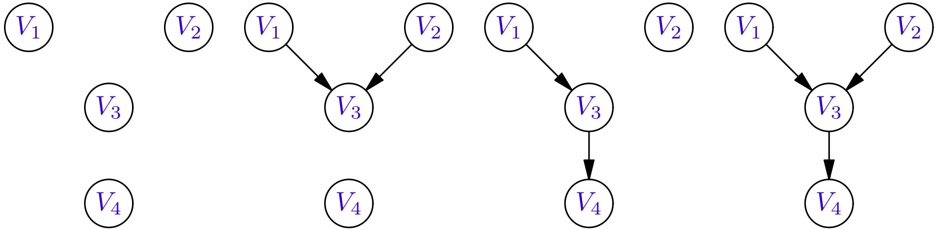
[3] i.e. directly connected by an arc

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$ defined by

$$I_{\text{Pr}}(\{V_1\}, \emptyset, \{V_2\}) \text{ and } I_{\text{Pr}}(\{V_1, V_2\}, \{V_3\}, \{V_4\})$$

Which of the following DAGs are D-maps for I_{Pr} ?



See Exercise 3.8 (Syllabus) Answer: all

Directed I-maps: what can they tell ?

Let I_{Pr} be an independence relation and G a DAG.

Consider an I-map for I_{Pr}, then

V_1 and V_2 non-neighbours $\Longrightarrow$ V_1, V_2 (cond.) independent,
or possibly(!) induced:
conditionally dependent

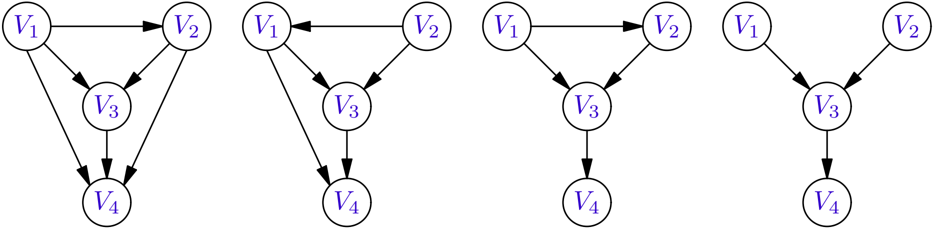
V_1 and V_2 neighbours $\Longrightarrow$?? V_1 and V_2 can be:
dependent,
independent,
conditionally dependent, *or*
conditionally independent

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$ defined by

$$I_{\mathrm{Pr}}(\{V_1\}, \emptyset, \{V_2\}) \text{ and } I_{\mathrm{Pr}}(\{V_1, V_2\}, \{V_3\}, \{V_4\})$$

Which of the following DAGs are I-maps for I_{Pr} ?



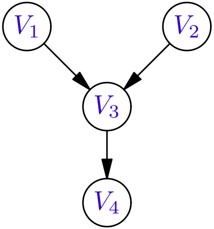
(See Exercise 3.8) Answer: all

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$ defined by

$$I_{\mathrm{Pr}}(\{V_1\}, \emptyset, \{V_2\}) \text{ and } I_{\mathrm{Pr}}(\{V_1, V_2\}, \{V_3\}, \{V_4\})$$

The following DAG is a perfect map for I_{Pr}:



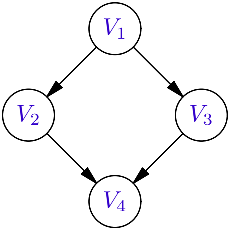
Is this P-map for I_{Pr} unique ? yes

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$ defined by

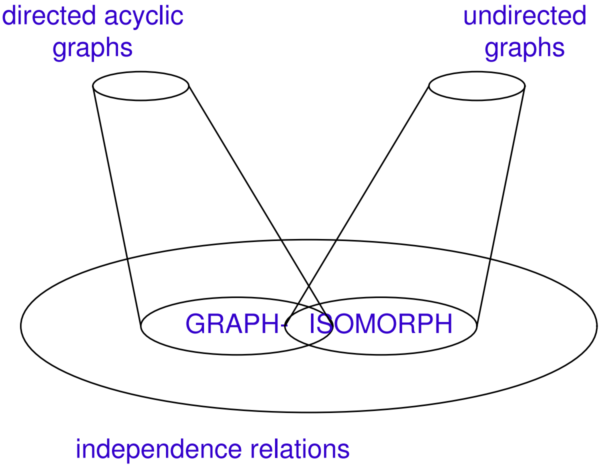
$$I_{\mathrm{Pr}}(\{V_1\}, \{V_2, V_3\}, \{V_4\}) \text{ and } I_{\mathrm{Pr}}(\{V_2\}, \{V_1, V_4\}, \{V_3\})$$

The relation I_{Pr} has an undirected P-map, but does not have a directed P-map. Consider for example the following DAG G:



In graph G we have that $\langle \{V_1\} \mid \{V_2, V_3\} \mid \{V_4\} \rangle_G^d$, but also that $\langle \{V_2\} \mid \{V_1\} \mid \{V_3\} \rangle_G^d$!

Independence relations and their graphical representation



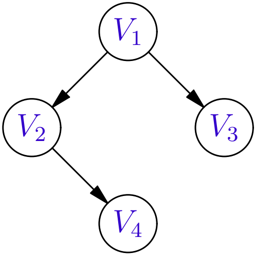
(Graph-isomorph: independence relation with perfect map.)

An I-map or a D-map ?

Reconsider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$ defined by

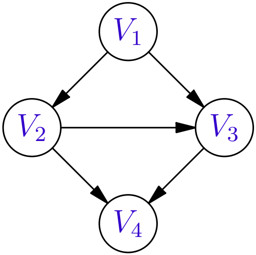
$$I_{\mathrm{Pr}}(\{V_1\}, \{V_2, V_3\}, \{V_4\}) \text{ and } I_{\mathrm{Pr}}(\{V_2\}, \{V_1, V_4\}, \{V_3\})$$

Compare the following two representations of independence relation I_{Pr}:



a D-map

and



an I-map

Recall what we were looking for...

- We can factorise any joint distribution using the chain rule:

$$\Pr(\boldsymbol{V}) \quad = \quad \Pr(V_n \mid V_{n-1} \wedge \ldots \wedge V_1) \cdot \ldots \cdot \Pr(V_2 \mid V_1) \cdot \Pr(V_1)$$

- We want to exploit $I_{\Pr}$ to factorise the joint more efficiently $\rightarrow$ store (conditional) distributions involving less variables:

$$\Pr(\boldsymbol{V}) \quad \stackrel{?}{=} \quad \Pr(V_n \mid V_m \wedge \ldots \wedge V_k) \cdot \ldots \cdot \Pr(V_2) \cdot \Pr(V_1)$$

BUT:

- $\Pr(X \mid Y) = \Pr(X)$ is mathematically correct only if X is truly independent of Y

A minimal I-map

Definition: Let I_{Pr} be an independence relation on a set of random variables $\boldsymbol{V}$. Let $G = (\boldsymbol{V}_G, \boldsymbol{A}_G)$ be a graph with $\boldsymbol{V}_G = \boldsymbol{V}$.

The graph G is called a minimal I-map for I_{Pr} if the following conditions hold:

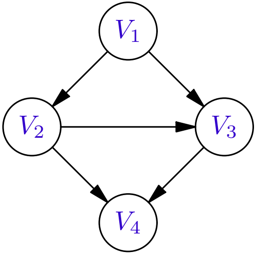
- G is an I-map for I_{Pr}, and
- no proper subgraph of G is an I-map for I_{Pr}.

An example

Consider the independence relation I_{Pr} on $\boldsymbol{V} = \{V_1, \ldots, V_4\}$ defined by

$$I_{\mathrm{Pr}}(\{V_1\}, \{V_2, V_3\}, \{V_4\}) \text{ and } I_{\mathrm{Pr}}(\{V_2\}, \{V_1, V_4\}, \{V_3\})$$

The following DAG is a minimal I-map for I_{Pr}:



Is this minimal I-map for I_{Pr} unique ?

no

Directed or undirected ? (I)

Directed and undirected I-maps are related.

Definition: The moral graph of a DAG $G = (\boldsymbol{V}_G, \boldsymbol{A}_G)$ is the undirected graph obtained as follows:

- for each $V_k \in \boldsymbol{V}_G$ add an edge between each pair of unconnected parents $V_i, V_j \in \boldsymbol{\rho}_G(V_k)$;
- drop the directions of all arcs.

Definition: A graph is triangulated or chordal if any loop of length ≥ 4 contains a shortcut.

Proposition: Let I be an independence relation over $\boldsymbol{V}$. Consider graphs $G = (\boldsymbol{V}_G, \boldsymbol{A}_G)$ and $G' = (\boldsymbol{V}, \boldsymbol{E}_{G'})$. Then,

$$G \text{ is an I-map for } I \quad \underset{\text{triangulation+add direction}}{\overset{\text{moralisation+drop direction}}{\rightleftarrows}} \quad G' \text{ is an I-map for } I$$

Directed or undirected ? (II)

Consider the following properties (partly proven later):

- Let G be a directed acyclic graph. Then G is a directed I-map of $I_{\Pr} \iff \Pr$ can be written as

$$\Pr(\boldsymbol{V}) = \prod_{V_i} \Pr(V_i \mid \boldsymbol{\rho}_G(V_i))$$

- Let G be an undirected graph. Then G is an undirected I-map of $I_{\Pr} \iff$[4] $\Pr$ can be written as

$$\Pr(\boldsymbol{V}) = K \cdot \prod_{\boldsymbol{C}_i} \Phi(\boldsymbol{C}_i) \quad \longleftarrow$$

what's the meaning of these clique potentials?!?

for some normalisation factor K.

[4] $\Longrightarrow$ requires $\Pr$ to be strictly positive